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Asymptotic stability in general systems

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I. INTRODUCTION

In his classic memoir, Lyapunov [7] introduced definitions of stability and asymptotic stability of solutions of ordinary differential equations and used certain norm-like functions in order to investigate questions raised by his definitions. We will very briefly outline this technique now known as Lyapunov's second method.

Let R denote the real numbers, R^n denote Euclidean n-dimensional space and $\|\cdot\|$ any norm on R^n. Let $x \in R^n$ and $F:R^n \times R \to R^n$ be continuous on $C(b,t_o) = \{(x,t) \in R^n \times R : \|x\| < b < \infty,\ t \geqq t_o\}$ and suppose that $F(0,t) = 0$ for all $t \geqq t_o$. Consider the equation

$$(1\text{-}1) \qquad \frac{dx}{dt} = F(x,t)$$

In this case $x(t) = 0,\ t \geqq t_o$ is a solution of (1-1). We will call it the null solution.

<u>Definition 1.1</u>. The null solution $x(t) = 0,\ t \geqq t_o$ of (1-1) is said to be stable at $t = t_o$ in case for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon,t_o) > 0$ such that whenever $\|x_o\| < \delta$, we have $\|x(t,t_o,x_o)\| < \epsilon$ for all $t \geqq t_o$ where $x(t,t_o,x_o)$

is a solution of (1-1) such that $x(t_o, t_o, x_o) = x_o$.

Definition 1.2. The null solution of (1-1) is said to be asymptotically stable at $t = t_o$ provided that it is stable at $t = t_o$ and that there is a $\delta_2(t_o) > 0$ such that if $\|x_o\| < \delta_2$ then $\lim_{t\to\infty} x(t, t_o, x_o) = 0$ where $x(t, t_o, x_o)$ is a solution of (1-1) such that $x(t_o, t_o, x_o) = x_o$.

Definition 1.3. Let $S(b_1) = \{x \in R^n : \|x\| \leqq b_1 < b\}$ and let $W: S(b_1) \to R$ such that $W(0) = 0$. $W(x)$ is called positive (negative) definite on $S(b_1)$ in case $W(x) > 0$ $(W(x) < 0)$ for all $x \in S(b_1)\setminus\{0\}$.

Definition 1.4. Let $C(b_1, T) = S(b_1) \times [T, \infty)$, $T \geqq t_o$ and let $V: C(b_1, T) \to R$ be such that $V(0, t) = 0$ for all $t \geqq T$. $V(x, t)$ is called positive (negative) semidefinite on $C(b_1, T)$ in case $V(x, t) \geqq 0$ $(V(x, t) \leqq 0)$ for all $(x, t) \in C(b_1, T)$. $V(x, t)$ is called positive (negative) definite on $C(b_1, T)$ in case there exists a positive (negative) definite function $W: S(b_1) \to R$ such that $V(x, t) \geqq W(x)$ $(V(x, t) \leqq W(x))$ for all $(x, t) \in C(b_1, T)$.

Definition 1.5. A function $f: C(b_1, T) \to R$ is said to have infinitesimal upper bound in case given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x, t)| < \epsilon$ whenever $\|x\| < \delta$ for all

$t \geqq T$.

Let $V: C(b_1,T) \to R$ and suppose that V has continuous first partial derivatives on $C(b_1,T)$, $V(0,t) = 0$, and that $V(x,t)$ is positive (negative) definite on $C(b_1,T)$.

Definition 1.6. $V(x,t)$ is called a Lyapunov function for (1-1) in case the function $\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i(x,t) + \frac{\partial V}{\partial t}$, which is the derivative of $V(x,t)$ along a solution of (1-1), is negative (positive) semidefinite on $C(b_1,T)$.

Theorem 1.7. (Lyapunov [7]). If there exists a Lyapunov function for (1-1) then the null solution of (1-1) is stable at t_o.

Theorem 1.8. (Lyapunov [7]). If there exists a function $V: C(b_1,T) \to R$ such that $V(x,t)$ has continuous first partial derivatives on $C(b_1,T)$, $V(0,t) = 0$, $V(x,t)$ is positive (negative) definite on $C(b_1,T)$,
$\frac{dV}{dt} = \sum_{k=1}^{n} \frac{\partial V}{\partial x_i} F_i(x,t) + \frac{\partial V}{\partial t}$ is negative (positive) definite on $C(b_1,T)$, and $V(x,t)$ has infinitesimal upper bound then the null solution of (1-1) is asymptotically stable.

There are partial converses to these two theorems. For a thorough exposition of Lyapunov's second method, we refer the reader to Yoshizawa [14].

Lyapunov's second method has been the subject of numerous investigations and modifications; in fact, now there is an extensive body of material dealing not only with stability and asymptotic stability of solutions of ordinary differential equations but more generally with stability and asymptotically in dynamical systems. ([1], [6], [12] and [14] provide a sampling of such material.) Until recently, the range of the so-called Lyapunov function has been restricted to be the real numbers, and as a consequence, the type of systems to which Lyapunov's second method could be applied has been restricted.

However, Bushaw [3] introduced the concept of the retracted scale of a filter of entourages of a uniformity and was able to give necessary and sufficient conditions for uniform stability in very general systems in terms of the existence of generalized Lyapunov functions which take their values in the retracted scale of a uniformity.

In this dissertation, it is our goal to formulate an asymptotic stability criterion for a general system defined on a uniform space and to give necessary and sufficient conditions in order that it be satisfied. The particular system with which we work is called a system for cones defined on a uniform space. This system is less general than those with which Bushaw has dealt but it appears that one must be prepared to sacrifice some degree of generality in order to be able to investigate systems for which an asymptotic stability concept may be defined.

In Chapter II we give those definitions and properties of filters, uniform spaces and retracted scales of uniformities which will be used throughout the dissertation. Since uniform spaces and uniform structures are not widely studied, particularly by applied mathematicians, some of the elementary properties are established.

In Chapter III we define the system with which we will work and what is meant by stability and asymptotic stability with respect to this system. We define a generalized Lyapunov function with infinitesimal upper bound and use it to give necessary and sufficient conditions for stability and asymptotic stability with respect to our system.

In Chapter IV we pursue a line of investigation which allows us to determine the continuity properties of our generalized Lyapunov function and discuss the extent to which continuity of our Lyapunov function is dependent on continuity properties of our system.

Finally, in Chapter V we show that a generalized dynamical system (or generalized control system) (see Roxin [9], [10] and [11]) defines a system of cones and show how our criteria may be applied.

II. PRELIMINARIES

In this chapter we introduce concepts which will be basic in our subsequent investigation. Our treatment is brief and we refer the reader to Bourbaki [2] or Kelley [5] for a thorough exposition on the topic of uniform spaces.

A. Filters and Uniformities

Throughout this section, X will denote a nonempty set.

Definition 2.1. A prefilter on X is a nonempty collection $\mathcal{A}$ of subsets of X satisfying:

F_1) The empty set does not belong to $\mathcal{A}$.

F_2) Every subset of X which contains a member of $\mathcal{A}$, belongs to $\mathcal{A}$.

Definition 2.2. A filter on X is a prefilter $\mathcal{F}$ on X which satisfies: F_3) The intersection of two members of $\mathcal{F}$ belongs to $\mathcal{F}$.

A uniform structure or uniformity on X is a collection of subsets of the Cartesian product $X \times X$, which is not only a filter but also satisfies certain other properties. Before we can formalize this notion we will have to introduce some more concepts.

<u>Definition 2.3</u>. The <u>diagonal subset</u>, $\Delta(X)$, of $X \times X$ is defined by

$$\Delta(X) = \{(x,x) : x \in X\}$$

<u>Definition 2.4</u>. If A is a subset of $X \times X$ then

$$A^{-1} = \{(y,x) : (x,y) \in A\}$$

<u>Definition 2.5</u>. If A and B are subsets of $X \times X$ then the <u>composition</u> of A and B is denoted by $A \circ B$ and is defined by:

$$A \circ B = \{(x,y) : (x,z) \in B \ (z,y) \in A \ \text{for some} \ z \in X\}$$

<u>Definition 2.6</u>. A <u>uniform structure</u> or <u>uniformity</u> $\mathcal{U}$ on X is a filter $\mathcal{U}$ on $X \times X$ which satisfies:

U_1) $\Delta(X) \subset U$ for all $U \in \mathcal{U}$.

U_2) If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.

U_3) If $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The pair $(X,\mathcal{U})$ is called a <u>uniform space</u>. The elements of $\mathcal{U}$ are called <u>entourages</u>.

The sets $U \in \mathcal{U}$ may be used to define a topology for X. For $x \in X$ and $U \in \mathcal{U}$, let $U[x] = \{y \in X : (x,y) \in U\}$.

<u>Definition 2.7</u>. The topology $\mathcal{J}$ on X generated by $\mathcal{U}$ is the collection of all subsets T of X satisfying: $T \in \mathcal{J}$

if and only if for each $x \in T$ there exists a $U \in \mathcal{U}$ such that $U[x] \subset T$. This topology is called the <u>uniform topology</u> for X.

In the sequel, it will be implicit that whenever we speak of a uniform space $(X,\mathcal{U})$, we consider it to be a topological space with the uniform topology.

One of the questions which we will consider concerns the continuity of a function f on a uniform space $(X,\mathcal{U})$ having values in a uniform space $(Y,\mathcal{Y})$. As usual f will be said to be continuous at $x_o \in X$ in case for each neighborhood N of $f(x_o)$ in the uniform topology generated on Y by $\mathcal{Y}$ we have that $f^{-1}(N)$ is a neighborhood of x_o in the uniform topology generated on X by $\mathcal{U}$. We may couch this in terms of members of the respective uniformities in the following fashion: f is continuous at $x_o \in X$ if and only if for each $V \in \mathcal{Y}$ there is a $U \in \mathcal{U}$ such that $f(y) \in V[f(x_o)]$ whenever $y \in U[x_o]$. We can also consider a stronger type of continuity which is elegantly described in uniform spaces.

<u>Definition 2.8</u>. $f:(X,\mathcal{U}) \to (Y,\mathcal{Y})$ is <u>uniformly continuous</u> relative to the uniformities $\mathcal{U}$ and $\mathcal{Y}$ if and only if for each $V \in \mathcal{Y}$ we have $\{(x,y) : (f(x),f(y)) \in V\} \in \mathcal{U}$.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniformities for X then we say that $\mathcal{U}_1$ is <u>coarser</u> than $\mathcal{U}_2$ in case $\mathcal{U}_1 \subset \mathcal{U}_2$.

Let A be an index set and suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{U}_\alpha)$ is a uniform space. We will denote the Cartesian product of the sets X_α, $\alpha \in A$, by $\prod_{\alpha \in A} X_\alpha$. (We recall that $\prod_{\alpha \in A} X_\alpha = \{x : A \to \bigcup_{\alpha \in A} X_\alpha : x(\alpha) = x_\alpha \in X_\alpha\}$.). The projection mapping onto the $\beta^{\underline{th}}$ coordinate $(\beta \in A)$ will be denoted by $pr_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$.

<u>Definition 2.9</u>. The <u>product</u> <u>uniformity</u> for $\prod_{\alpha \in A} X_\alpha$ is the coarsest uniformity such that each projection, pr_β, $\beta \in A$, is a uniformly continuous function relative to the product uniformity and $\mathcal{U}_\beta$.

It is easy to show that the topology generated on $\prod_{\alpha \in A} X_\alpha$ by the product uniformity is the usual product topology.

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a <u>base</u> for $\mathcal{U}$ in case each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subfamily $\mathcal{S}$ of a uniformity $\mathcal{U}$ is a <u>subbase</u> for $\mathcal{U}$ in case the collection of finite intersections of members of $\mathcal{S}$ constitutes a base for $\mathcal{U}$.

If we let $\sigma(U_\alpha) = pr_\alpha^{-1}(U_\alpha)$ for $U_\alpha \in \mathcal{U}_\alpha$, $\alpha \in A$, then it is easy to show that the collection:

$$\mathcal{S} = \{\sigma(U_\alpha) : U_\alpha \in \mathcal{U}_\alpha, \alpha \in A\}$$

is a subbase for the product uniformity on $\prod_{\alpha \in A} X_\alpha$.

Definition 2.10. Let (M,d) be a metric space and let

$$V_{d,\epsilon} = \{(x,y) \in M \times M : d(x,y) < \epsilon\} \quad \text{for} \quad \epsilon > 0.$$

The metric uniformity on M is the uniformity $\mathcal{M}$ whose base is the collection $\mathcal{B} = \{V_{d,\epsilon} : \epsilon > 0\}$.

It is easy to show that the uniform topology generated on M by $\mathcal{M}$ is the same as the metric topology on M.

In what follows we will deal with an arbitrary uniform space $(X,\mathcal{U})$ and the real numbers equipped with the metric uniformity which we will denote by $(R,\mathcal{R})$. (We take as a metric on R, the function $d(x,y) = |x - y|$.) We will also have to consider two product spaces $(X \times R,\mathcal{W})$ and $(X \times R \times R,\mathcal{Z})$ where $\mathcal{W}$ and $\mathcal{Z}$ denote the product uniformities on $X \times R$ and $X \times R \times R$ respectively. Since there are only a finite number of factors in the product spaces we wish to consider, we can actually characterize the base elements for the product uniformities $\mathcal{W}$ and $\mathcal{Z}$ in a simple manner.

Define $S(U,V) = \{((x,s),(y,t)) \in (X \times R) \times (X \times R) : (x,y) \in U, (s,t) \in V\}$ for each $U \in \mathcal{U}$ and $V \in \mathcal{R}$, and define for $U \in \mathcal{U}, V, V' \in \mathcal{R}$

$S(U,V,V') = \{((x,s,p),(y,t,q)) \in (X \times R \times R) \times (X \times R \times R) : (x,y) \in U, (s,t) \in V, (p,q) \in V'\}$. If $pr_i (i = 1,2,3)$ denotes the projection onto the $i^{\underline{th}}$ coordinate then it is easy to see that $S(U,V) = pr_1^{-1}(U) \cap pr_2^{-1}(V)$ and that $S(U,V,V') = pr_1^{-1}(U) \cap pr_2^{-1}(V) \cap pr_3^{-1}(V)$. Thus $S(U,V)$ for $U \in \mathcal{U}$, $V \in \mathcal{R}$ is a member of a base for $\mathcal{W}$ since it is the intersection of two elements of a subbase for $\mathcal{W}$. Also since every element $S(U,V)$ is a finite intersection of elements of a subbase for $\mathcal{W}$, it follows easily that the collection of elements of the form $S(U,V)$ for $U \in \mathcal{U}$, $V \in \mathcal{R}$ is a base for $\mathcal{W}$. Similarly, the collection of all sets of the form $S(U,V,V')$ for $U \in \mathcal{U}, V, V' \in \mathcal{R}$ forms a base for $\mathcal{Z}$. Thus we have the following:

<u>Lemma 2.11</u>. The collection $\{S(U,V) : U \in \mathcal{U}, V \in \mathcal{R}\}$ is a base for the product uniformity $\mathcal{W}$ on $X \times R$. The

collection $\{S(U,V,V') : U \in \mathcal{U}, V,V' \in \mathcal{R}\}$ is a base for the product uniformity $\mathcal{Z}$ on $X \times R \times R$.

B. The Retracted Scale of a Uniformity

Let $\mathcal{F}$ be a filter and denote by $\mathcal{P}(\mathcal{F})$ the collection of all prefilters contained in $\mathcal{F}$. We can partially order $\mathcal{P}(\mathcal{F})$ by defining $\alpha \leqq \beta$ for $\alpha,\beta \in \mathcal{P}(\mathcal{F})$ if and only if $\beta \subseteqq \alpha$. The partially ordered set $(\mathcal{P}(\mathcal{F}),\leqq)$ is called the <u>scale</u> of the filter $\mathcal{F}$. We will be interested in the scale of the filter of entourages $\mathcal{Y}$ of a uniform space $(Y,\mathcal{Y})$.

Let $\alpha \in \mathcal{P}(\mathcal{Y})$ and $U \in \mathcal{Y}$ then we will define

$$U\langle\alpha\rangle = \{U \circ V : V \in \alpha\}.$$

For $U \in \mathcal{Y}$ we will define

$$\tilde{U} = \{(\alpha,\beta) \in \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y}) : U\langle\alpha\rangle \subset \beta, U\langle\beta\rangle \subset \alpha\}.$$

Let $\tilde{\mathcal{Y}} = \{\mathcal{A} \subset \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y})$: there is a $U \in \mathcal{Y}$ such that $\tilde{U} \subset \mathcal{A}\}$.

<u>Proposition 2.12</u>. (Bushaw [3]) $\tilde{\mathcal{Y}}$ is a uniformity on $\mathcal{P}(\mathcal{Y})$ and the collection $\mathcal{B} = \{\tilde{U} : U \in \mathcal{Y}\}$ is a base for $\tilde{\mathcal{Y}}$.

For our purposes $\mathcal{P}(\mathcal{Y})$ will, in general, contain too many prefilters. However, the introduction of the uniformity $\tilde{\mathcal{U}}$ will allow us to move to a quotient space where things are less complicated.

Definition 2.13. A uniformity $\mathcal{U}$ on Y is said to be separated or Hausdorff in case $\cap\{U : U \in \mathcal{U}\} = \Delta(Y)$.

The origin of the term Hausdorff is clear when one considers the fact that the uniform topology on Y is T_2 (Hausdorff) if and only if $\mathcal{U}$ is separated.

We will define a relation on $\mathcal{P}(\mathcal{U})$ by: if $\alpha,\beta \in \mathcal{P}(\mathcal{U})$, $\alpha \sim \beta$ if and only if $(\alpha,\beta) \in \cap\{\mathcal{a} : \mathcal{a} \in \tilde{\mathcal{U}}\}$. It is not difficult to verify that this is an equivalence relation. We refer the reader to Bourbaki [2] for the details. Let $\hat{\alpha} = \{\beta \in \mathcal{P}(\mathcal{U}) : \beta \sim \alpha\}$ for each $\alpha \in \mathcal{P}(\mathcal{U})$ and $\widehat{\mathcal{P}(\mathcal{U})} = \{\hat{\alpha} : \alpha \in \mathcal{P}(\mathcal{U})\}$. Define $f:\mathcal{P}(\mathcal{U}) \to \widehat{\mathcal{P}(\mathcal{U})}$ by $f(\alpha) = \hat{\alpha}$ for each $\alpha \in \mathcal{P}(\mathcal{U})$. and $g:\mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U}) \to \widehat{\mathcal{P}(\mathcal{U})} \times \widehat{\mathcal{P}(\mathcal{U})}$ by $g(\alpha,\beta) = (f(\alpha),f(\beta))$. Let $\hat{\tilde{\mathcal{U}}} = \{\hat{\mathcal{a}} \in \widehat{\mathcal{P}(\mathcal{U})} \times \widehat{\mathcal{P}(\mathcal{U})} :$ there is an $\mathcal{a} \in \tilde{\mathcal{U}} \ni g(\mathcal{a}) \subset \hat{\mathcal{a}}\}$. It can be shown that $\hat{\tilde{\mathcal{U}}}$ is a uniformity on $\widehat{\mathcal{P}(\mathcal{U})}$ and what is more, $\hat{\tilde{\mathcal{U}}}$ is separated. The uniform space $(\widehat{\mathcal{P}(\mathcal{U})},\hat{\tilde{\mathcal{U}}})$ is called the associated separated space of $(\mathcal{P}(\mathcal{U}),\tilde{\mathcal{U}})$.

It is clear that from the standpoint of notation, passage to the associated separated space is rather awkward.

In [3], Bushaw introduced a subset of $\mathcal{P}(\mathcal{Y})$ which is effectively equivalent to $\mathcal{P}(\mathcal{Y})$ with the added advantage that it lends itself to easy application.

We define the mapping $w:\mathcal{P}(\mathcal{Y}) \to \mathcal{P}(\mathcal{Y})$ by

$$\text{(2-1)}\quad w(\alpha) = \{U \in \mathcal{Y} : V \circ U \in \alpha \text{ for all } V \in \mathcal{Y}\}$$
$$\text{for } \alpha \in \mathcal{P}(\mathcal{Y})$$

<u>Proposition 2.14</u>. (Bushaw [3]) For all $\alpha \in \mathcal{P}(\mathcal{Y})$, $w(\alpha) \leqq \alpha$ and $w(w(\alpha)) = w(\alpha)$. Furthermore, if $\alpha,\beta \in \mathcal{P}(\mathcal{Y})$ such that $\alpha \leqq \beta$ then $w(\alpha) \leqq w(\beta)$. In other words, w is decreasing relative to the order we have defined on $\mathcal{P}(\mathcal{Y})$, idempotent, and order-preserving.

<u>Proposition 2.15</u>. (Bushaw [3]) Suppose $\alpha,\beta \in \mathcal{P}(\mathcal{Y})$. Then $\alpha \sim \beta$ if and only if $w(\alpha) = w(\beta)$.

<u>Definition 2.16</u>. (Bushaw [3]) A prefilter $\alpha \in \mathcal{P}(\mathcal{Y})$ is said to be <u>regular</u> in case $w(\alpha) = \alpha$.

We will denote by $\mathcal{P}_r(\mathcal{Y})$ the collection of all regular prefilters in $\mathcal{P}(\mathcal{Y})$. The collection $\mathcal{P}_r(\mathcal{Y})$ together with the partial ordering $\leqq$ inherited from $(\mathcal{P}(\mathcal{Y}),\leqq)$ is called the <u>retracted</u> <u>scale</u> of $\mathcal{Y}$.

Remark 2.17. Since $w(w(\alpha)) = w(\alpha)$ for all $\alpha \in \mathcal{P}(\mathcal{Y})$ it is clear that $w(\alpha) \in \mathcal{P}_r(\mathcal{Y})$ for all $\alpha \in \mathcal{P}(\mathcal{Y})$. Hence the range of w is actually $\mathcal{P}_r(\mathcal{Y})$.

It is clear that $\mathcal{Y} \leqq \alpha$ for all $\alpha \in \mathcal{P}(\mathcal{Y})$. Thus plays the role of the smallest element of the scale of $\mathcal{Y}$. When we wish to work with the scale of $\mathcal{Y}$ and no confusion will arise, we will denote $\mathcal{Y}$ by 0. Using this new notation we state the following result.

Proposition 2.18. (Bushaw [3]) $w(\alpha) = 0$ if and only if $\alpha = 0$.

We may characterize the regular prefilters in a pleasing fashion by using the following result.

Proposition 2.19. $\alpha \in \mathcal{P}_r(\mathcal{Y})$ if and only if $\beta \sim \alpha$ implies that $\alpha \leqq \beta$.

Proof: Suppose $\alpha \in \mathcal{P}_r(\mathcal{Y})$ and that $\beta \in \mathcal{P}(\mathcal{Y})$ is such that $\beta \sim \alpha$. By Proposition 2.15 $w(\beta) = w(\alpha)$ and by Proposition 2.14 $w(\beta) \leqq \beta$. Now $\alpha = w(\alpha)$ by Definition 2.16 so that we have $\alpha = w(\alpha) = w(\beta) \leqq \beta$ or $\alpha \leqq \beta$.

Conversely suppose that $\alpha \in \mathcal{P}(\mathcal{Y})$ is such that $\beta \sim \alpha$ implies that $\alpha \leqq \beta$. From Proposition 2.14 $w(w(\alpha)) = w(\alpha)$ so that it follows by Proposition 2.15 that $w(\alpha) \sim \alpha$ and hence by the condition we have assumed $\alpha \leqq w(\alpha)$. But by Proposition 2.14 $w(\alpha) \leqq \alpha$, so by the antisymmetry of the partial order $\leqq$, it follows that $w(\alpha) = \alpha$ and hence $\alpha \in \mathcal{P}_r(\mathcal{Y})$.

As we noted in the preceding proof $w(\alpha) \sim \alpha$ for all $\alpha \in \mathcal{P}(\mathcal{Y})$. Thus, for each $\alpha \in \mathcal{P}(\mathcal{Y})$, the equivalence class $\hat{\alpha} \in \widehat{\mathcal{P}(\mathcal{Y})}$ contains at least one regular member, namely $w(\alpha)$. It follows then that a regular prefilter is the "smallest" member of its equivalence class relative to the partial ordering $\leqq$. Indeed, Bushaw [3] has shown that each equivalence class $\hat{\alpha} \in \widehat{\mathcal{P}(\mathcal{Y})}$ contains a unique regular member. Thus we can state the following result.

Proposition 2.20. There is a one-to-one correspondence between $\widehat{\mathcal{P}(\mathcal{Y})}$ and $\mathcal{P}_r(\mathcal{Y})$.

As a subset of $\mathcal{P}(\mathcal{Y})$, $\mathcal{P}_r(\mathcal{Y})$ inherits a uniformity given by $\tilde{\mathcal{Y}}_r = \{a \cap (\mathcal{P}_r(\mathcal{Y}) \times \mathcal{P}_r(\mathcal{Y})) : a \in \tilde{\mathcal{Y}}\}$.

Proposition 2.21. (Bushaw [3]) The mapping $w: \mathcal{P}(\mathcal{Y}) \to \mathcal{P}_r(\mathcal{Y})$ is uniformly continuous relative to the uniformities $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{Y}}_r$.

C. Some Properties of Uniformities.

Since the manipulations which we will be required to do are likely to be unfamiliar to the non-specialist, we present those which we will encounter frequently. Throughout this section $(X;\mathcal{U})$ will be an arbitrary uniform space. If A is a nonempty subset of X then for any $U \in \mathcal{U}$, $U[A] = \{y \in X : y \in U[a] \text{ for some } a \in A\}$.

Lemma 2.22. If $U_1, U_2 \in \mathcal{U}$ such that $U_1 \subset U_2$ and A and B are non-empty subsets of X such that $A \subset B$ then $U_1[A] \subset U_1[B] \subset U_2[B]$.

Proof: Suppose $y \in U_1[A]$ then for some $a \in A$ $y \in U_1[a]$. However, $A \subset B$ by hypothesis so that $y \in U_1[a]$ for some $a \in B$ and hence $y \in U_1[B]$. Thus $U_1[A] \subset U_1[B]$.

Now suppose that $y \in U_1[B]$ then for some $b \in B$, $y \in U_1[b]$ or $(b,y) \in U_1$. But $U_1 \subset U_2$ so that $(b,y) \in U_2$ and hence $y \in U_2[b]$ which in turn implies $y \in U_2[B]$. Thus $U_1[B] \subset U_2[B]$.

Lemma 2.23. If $U_1, U_2 \in \mathcal{U}$ and A is a nonempty subset of X then $U_1[U_2[A]] = U_1 \circ U_2[A]$.

Proof: Suppose $y \in U_1 \circ U_2[A]$. By definition there is an $a \in A$ such that $(a,y) \in U_1 \circ U_2$. This, in turn, is equivalent to the statement that there exists a $b \in X$ such that $(a,b) \in U_2$ and $(b,y) \in U_1$, or $b \in U_2[a]$ and

$y \in U_1[b]$. Hence, by definition, $y \in U_1[b]$ and $b \in U_2[A]$ so that $y \in U_1[U_2[A]]$.

The closure of a nonempty subset A of X in the uniform topology has a particularly useful representation in terms of the members of $\mathcal{U}$.

Lemma 2.24. For any nonempty subset A of X, $\bar{A} = \cap \{U[A] : U \in \mathcal{U}\}$. (Closure in the uniform topology).

Proof: If $x \in \bar{A}$ then for any $T \in \mathcal{J}$ (the uniform topology) such that $x \in T$ we have $T \cap A \neq \emptyset$. In particular the sets $\operatorname{int} U[x] = \{y \in U[x]$: there is a $V \in \mathcal{U}$ such that $V[y] \subset U[x]\} \in \mathcal{J}$ for all $U \in \mathcal{U}$. Thus, for any $U \in \mathcal{U}$ there is an $a_{U^{-1}} \in A$ such that $a_{U^{-1}} \in U^{-1}[x] \cap A$. Hence $a_{U^{-1}} \in U^{-1}[x]$ which is equivalent to $(x, a_{U^{-1}}) \in U^{-1}$ or $(a_{U^{-1}}, x) \in U$. It then follows that $x \in U[a_{U^{-1}}] \subset U[A]$ and we have $\bar{A} \subset \cap \{U[A] : U \in \mathcal{U}\}$.

Conversely, if $x \notin \bar{A}$ then for some $T \in \mathcal{J}$ such that $x \in T$ we must have $T \cap A = \emptyset$. But be definition of $\mathcal{J}$ there is a $V \in \mathcal{U}$ such that $V[x] \subset T$ and thus $V[x] \cap A = \emptyset$. We claim that $x \notin V^{-1}[A]$. If $x \in V^{-1}[A]$ there is an $a \in A$ such that $(a,x) \in V^{-1}$ which in turn gives us that $(x,a) \in V$ so that $a \in V[x]$ and we would have $V[x] \cap A \neq \emptyset$, a contradiction. But $x \notin V^{-1}[A]$ implies $x \notin \cap \{U[A] : U \in \mathcal{U}\}$ and it follows that $x \notin \bar{A}$ implies $x \notin \cap \{U[A] : U \in \mathcal{U}\}$ or, contrapositively, $\cap \{U[A] : U \in \mathcal{U}\} \subset \bar{A}$ and we are done.

Finally we state a result concerning the form of certain sets in the product space $(X \times R; \mathcal{W})$.

Lemma 2.25. If $A \subset X \times R$ is of the form $B \times R$ where B is a nonempty subset of X, $U \in \mathcal{U}$, and $V \in \mathcal{R}$, then $S(U,V)[A] = U[B] \times R$.

Proof: Let $(x,t) \in U[B] \times R$, then $x \in U[B]$ so that there is an $x_1 \in B$ such that $x \in U[x_1]$ or $(x_1,x) \in U$. Since $(t,t) \in V$ for all $t \in R$ we have $((x_1,t),(x,t)) \in S(U,V)$. But $(x_1,t) \in B \times R = A$ so that $(x,t) \in S(U,V)[A]$.

On the other hand if $(x,t) \in S(U,V)[A]$ there exists an $(x_1,t_1) \in A$ such that $(x,t) \in S(U,V)[(x_1,t_1)]$ or $((x_1,t_1),(x,t)) \in S(U,V)$. Thus we have, by definition, $(x_1,x) \in U$ and $(t_1,t) \in V$. From the form of A it follows that $x_1 \in B$ and, hence, we have $x \in U[B]$. Clearly then $(x,t) \in U[B] \times R$ and we are done.

III. STABILITY AND ASYMPTOTIC STABILITY

A. Systems of Cones on $X \times R$

In this chapter we will deal exclusively with the uniform space $(X \times R;\mathcal{W})$ introduced in Chapter II. We will use the notation 2^Y to denote the collection of all subsets of a set Y.

Definition 3.1. The mapping $s_\tau : 2^{X\times R} \to 2^X$, $\tau \in R$, is defined by

$$s_\tau(A) = \{x \in X : (x,\tau) \in A\} \quad \text{for} \quad A \in 2^{X\times R}$$

It is our aim to establish an asymptotic stability criterion for general systems. In order to realize this goal we introduce the concept of a system of cones on $X \times R$. We make one further notational convention by setting $p_\alpha = (x_\alpha, t_\alpha)$ for $(x_\alpha, t_\alpha) \in X \times R$.

Definition 3.2. Let a mapping $F: X \times R \to 2^{X\times R}$ satisfy:

C_1) $s_{t_o}(F(p_o)) = \{x_o\}$ for all $p_o \in X \times R$

C_2) $s_\tau(F(p_o)) = \emptyset$ if $\tau < t_o$

$s_\tau(F(p_o)) \neq \emptyset$ if $\tau \geqq t_o$ for all $p_o \in X \times R$

c_3) If $p_1 \in F(p_o)$ then $F(p_1) \subset F(p_o)$ for all $p_o \in X \times R$

The set $F(p_o)$ is called the <u>cone on</u> p_o <u>defined by</u> <u>F</u> or more simply the <u>cone on</u> p_o. The collection $\wp(F) = \{F(p) : p \in X \times R\}$ is called the <u>system of cones on</u> $X \times R$ <u>defined by</u> F.

A cone on p_o may be interpreted in the following manner: If the points $x \in X$ represent the possible states for some process and the point $p_\alpha \in X \times R$ represents the event that the process is in state x_α at time t_α, then $F(p_o)$ consists of p_o together with all events which may follow p_o in the evolution of the process. In other words, $F(p_o)$ consists of p_o and all events "attainable" from p_o in the course of the process. In this latter terminology, $s_\tau(F(p_o)) \times \{\tau\}$, $\tau \geqq t_o$, is the set of events attainable from p_o at time τ and $s_\tau(F(p_o))$, $\tau \geqq t_o$ is the set of states attainable $\tau - t_o$ time units after the process is in state x_o.

<u>Example 3.3</u>. Let $(X,\mathcal{U})$ be an arbitrary uniform space and define $K: X \times R \to 2^{X \times R}$ by:

$$K(p_o) = \{(x_o,t) : t \geqq t_o\} \text{ for } p_o = (x_o,t_o) \in X \times R$$

$\beta(K) = \{K(p) : p \in X \times R\}$ is easily shown to be a system of cones on $X \times R$.

Example 3.4. A more interesting example of a system of cones is that which is defined by the solutions of an ordinary differential equation. Let X by R^n and $\mathcal{U}$ be the metric uniformity on R^n. The product uniformity $\mathcal{W}$ on $X \times R$ is, in this case, the same as the metric uniformity on R^{n+1}.

Let $A(t)$ be an $n \times n$ matrix function whose elements are continuous and bounded for $t \in R$, let $b(t)$ be a n-vector whose components are continuous and bounded for $t \in R$, and let $x \in R^n$. Consider

$$(3\text{-}1) \qquad \frac{dx}{dt} = A(t)x + b(t).$$

Let $X(t)$ be a fundamental matrix solution of the associated homogeneous linear equation,

$$(3\text{-}2) \qquad \frac{dx}{dt} = A(t)x$$

which satisfies $X(0) = I_{n \times n}$ (where $I_{n \times n}$ is the $n \times n$ identity matrix). The (unique) solution of (3-1) through the point $p_o = (x_o, t_o) \in R^n \times R$ is given by:

$$(3\text{-}3) \quad x(t,p_o) = X(t)\{X^{-1}(t_o)x_o + \int_{t_o}^{t} X^{-1}(s)b(s)ds\} \quad t \in R$$

(See Coddington and Levinson [4] Chapter 3).

Let $F:R^{n+1} \to 2^{R^{n+1}}$ be defined by:

$$F(p_o) = \{(x(t,p_o),t) : x(t,p_o) \text{ is given by (3-3)}, \ t \geqq t_o\}$$

Clearly $F(p_o)$ is a cone on p_o since C_1) and C_2) are satisfied. The collection $\mathcal{G}(F) = \{F(p) : p \in X \times R\}$ is a system of cones on R^{n+1} for if $p_1 \in F(p_o)$ we have $F(p_1) = \{(x(t,p_1),t) \ t \geqq t_1\}$. But by uniqueness of solutions of (3-1) we have $x(t,p_1) = x(t,p_o)$ for $t \geqq t_1$ thus $F(p_1) = \{(x(t,p_o),t) \ t \geqq t_1\} \subset F(p_o)$.

Remark 3.5. The preceding example uses the uniqueness of solutions of (3-1) to show that F defines a system of cones on R^{n+1}. Uniqueness of solutions is not necessary; it is the fact that solutions of (3-1) through any $p_o = (x_o,t_o) \in R^{n+1}$ exist for all $t \geqq t_o$ which is crucial.

Example 3.6. As another example of a system of cones defined on R^{n+1} by an ordinary differential equation, consider the following. Suppose $f:R^n \times R \to R^n$ is a continuous function such that $f(\bar{0},t) = \bar{0}$, $\bar{0} \in R^n$, and that for some neighborhood N of $\bar{0}$ in R^n

$$(3\text{-}4) \qquad \frac{dx}{dt} = f(x,t),\ x(t_o) = x_o$$

has a unique solution which exists for all $t \geqq t_o$ for each $x_o \in N$. We will define a system of cones on R^{n+1} by $G:R^{n+1} \to 2^{R^{n+1}}$ given as follows:

$$G(p_o) = \{(x(t),t)\ t \geqq t_o,\ x(t) \text{ a solution of } (3\text{-}4)\}$$

if $p_o \in N \times \{t_o\}$.

Let $A = \bigcup_{p \in N \times \{t_o\}} G(p)$ and define the remainder of the cones of the system by

$$G(p_1) = \{p_1\} \cup \{(\bar{0},t) : t > t_1\} \text{ if } p_1 \in R^{n+1} \backslash A$$

and if $p_2 \in A \backslash N \times \{t_o\}$ then $p_2 \in G(p_o)$ for exactly one $p_o \in N \times \{t_o\}$ so we define

$$G(p_2) = \{(x,t) \in G(p_o) : t \geqq t_2\}.$$

The importance of the system of cones $\mathscr{C}(G)$ defined by G lies in the fact that it is the system of cones which we would use if we were to apply the methods developed in this dissertation to the investigation of the classical asymptotic stability at t_o of the null solution of (3-4). Again uniqueness of solutions is used but we reiterate that it is not a crucial consideration.

Definition 3.7. Let $F(p_o)$ be a cone on $p_o \in X \times R$. The τ-tail of $F(p_o)$ is the set $\{(x,t) \in F(p_o) : t \geqq \tau\}$ and will be denoted by $F_\tau(p_o)$.
Note that if $\tau \leqq t_o$ then we have $F_\tau(p_o) = F(p_o)$ and that if $\tau > t_o$ then $F_\tau(p_o) \subset F(p_o)$.

Remark 3.8. A system of cones $\mathscr{C}(F)$ defined by F on $X \times R$ imposes a natural quasi-order (that is a reflexive, transitive binary relation) on $X \times R$. This natural quasi-order, which we will denote by ρ, is defined by:

$$(3\text{-}5) \qquad (p_o,p_1) \in \rho \Longleftrightarrow p_1 \in F(p_o)$$

Clearly $(p_o,p_o) \in \rho$ because $p_o \in F(p_o)$ for all

$p_o \in X \times R$. Also if $(p_o,p_1) \in \rho$ $(p_1,p_2) \in \rho$ then by (3-5) $p_1 \in F(p_o)$ and $p_2 \in F(p_1)$ by c_3) $F(p_1) \subset F(p_o)$ so that $p_2 \in F(p_1) \subset F(p_o)$ and thus $(p_o,p_2) \in \rho$.

If we use the notation $\rho(p_o)$ to denote the collection of all $p_1 \in X \times R$ such that $(p_o,p_1) \in \rho$ we see that $\rho(p_o) = F(p_o)$, for if $p_1 \in \rho(p_o)$ then $(p_o,p_1) \in \rho$ so that $p_1 \in F(p_o)$ by (3-5). Conversely if $p_1 \in F(p_o)$ then (3-5) gives us that $(p_o,p_1) \in \rho$ and thus $p_1 \in \rho(p_o)$.

This remark is of fundamental importance in establishing a bridge between our work and that of Bushaw [3].

B. Stability and Asymptotic Stability of Cones with Respect to a System of Cones

In order to formulate an asymptotic stability concept we first need to define what we will mean by the term "approach". Let $(Y,\mathcal{U})$ be an arbitrary uniform space and R be the real numbers. Suppose $G:R \to 2^Y$ such that $G(t) \neq \emptyset$ for all $t \geqq t_o$ for some $t_o \in R$, and suppose that A is a nonempty subset of Y.

Definition 3.9. $G(t)$ is said to approach A, denoted by $G(t) \to A$, as $t \to \infty$ in case given $U \in \mathcal{U}$ there exists a $T(U) \in R$ such that $G(t) \subset U[A]$ whenever $t \geqq t_o + T(U)$.

We are now ready to define stability and asymptotic stability of a set with respect to a system of cones. Let $\mathcal{B}(F)$ be a system of cones defined on $X \times R$ by F.

<u>Definition 3.10</u>. A nonempty subset A of $X \times R$ is said to be stable with respect to the system of cones $\mathcal{B}(F)$ and uniformity $\mathcal{W}$, denoted "stable $(F,\mathcal{W})$", in case given $W \in \mathcal{W}$ there is a $V \in \mathcal{W}$ such that $F(p_o) \subset W[A]$ whenever $p_o \in V[A]$.

<u>Definition 3.11</u>. A non-empty subset A of $X \times R$ is said to be quasi-asymptotically stable with respect to the system of cones $\mathcal{B}(F)$ and uniformity $\mathcal{W}$, denoted "q.a.s.$(F,\mathcal{W})$", in case for each $p_o \in A$ there is a $V_{p_o} \in \mathcal{W}$ such that $F_t(p_1) \to A$ as $t \to \infty$ whenever $p_1 \in V_{p_o}[p_o]$. If the entourage V is independent of $p_o \in A$ then A is said to be uniformly quasi-asymptotically stable $(F,\mathcal{W})$ or "u.q.a.s.$(F,\mathcal{W})$".

<u>Definition 3.12</u>. A non-empty subset A of $X \times R$ is said to be asymptotically stable with respect to the system of cones $\mathcal{B}(F)$ and uniformity $\mathcal{W}$, denoted "a.s.$(F,\mathcal{W})$", in case A is both stable $(F,\mathcal{W})$ and q.a.s.$(F,\mathcal{W})$. A is uniformly asymptotically stable with respect to the system

of cones $\mathcal{B}(F)$ and uniformity $\mathcal{W}$, denoted "u.a.s.$(F;\mathcal{W})$", in case A is stable $(F;\mathcal{W})$ and u.q.a.s.$(F;\mathcal{W})$.

We give a few examples to clarify these ideas.

Example 3.13. Let $(X;\mathcal{U})$ and $\mathcal{B}(K)$ be as in example 3.3 and $x_o \in X$. The set $A_{x_o} = \{x_o\} \times R$ is stable $(K;\mathcal{W})$ but is not q.a.s.$(K;\mathcal{W})$ and hence, not a.s.$(K;\mathcal{W})$.

Example 3.14. Let $(X;\mathcal{U})$ and $\mathcal{B}(G)$ be as in example 3.6. If the null solution of (3-4) is [asymptotically] stable at t_o in the classical sense (see definitions 1.1 and 1.2) then the set $A_o = \{(\bar{0},t)\ t \in R,\ \bar{0} \in R^n\}$ is [a.s.] stable $(G;\mathcal{W})$ and conversely.

Example 3.15. Let $X = R$ and $\mathcal{U}$ be the metric uniformity on R. We define a system of cones on $R \times R$ in the following fashion: $(p_o = (x_o,t_o))$

1. $F(p_o) = \{(x,t) : x = x_o,\ t \geqq t_o\}$ if
   $x_o < -1,\ t_o \in R$ or $x_o > 1,\ t_o \in R$ or
   $t_o > 0,\ e^{-t_o} < x_o \leqq 1$ or $t_o > 0,\ -1 \leqq x_o < -e^{-t_o}$.
2. $F(p_o) = \{(x,t) : x = 1,\ t_o \leqq t \leqq 0\} \cup$
   $\{(x,t) : x = e^{-t},\ t \geqq 0\}$ if $x_o = 1,\ t_o < 0$.
3. $F(p_o) = \{(x,t) : x = -1,\ t_o \leqq t \leqq 0\} \cup$
   $\{(x,t) : x = -e^{-t},\ t \geqq 0\}$ if $x_o = -1$,

$t_o < 0$.

4. $F(p_o) = \{(x,t) : x = x_o,\ t_o \leqq t \leqq -\ln(-x_o)\} \cup$

$\{(x,t) : x = -e^{-t},\ t \geqq -\ln(-x_o)\}$ if

$-1 < x_o < 0,\ t_o \leqq -\ln(-x_o)$.

5. $F(p_o) = \{(x,t) : x = x_o,\ t_o \leqq t \leqq -\ln x_o\} \cup$

$\{(x,t) : x = e^{-t},\ t \geqq -\ln x_o\}$ if

$0 < x_o < e^{-t_o},\ t_o \geqq 0$.

6. $F(p_o) = \{(x,t) : x = x_o,\ t_o \leqq t \leqq -\ln x_o\} \cup$

$\{(x,t) : x = e^{-t}\ t \geqq -\ln x_o\}$ if

$0 < x_o < t_o + 1,\ -1 < t_o \leqq 0$.

7. $F(p_o) = \{(x,t) : x = 0\ t \geqq t_o\}$ if $x_o = 0,\ t_o > -1$.

8. $F(p_o) = \{(x,t) : x = t + x_o - t_o,$

$t_o \leqq t \leqq 1 - x_o + t_1\} \cup$

$\{(x,t) : x = 1,\ 1 - x_o + t_o \leqq t \leqq 0\} \cup$

$\{(x,t) : x = e^{-t}\ t \geqq 0\}$ if $0 \leqq x_o < 1$,

$t_o \leqq -1$.

The set $A_1 = \{(0,t) : t \in R\}$ is q.a.s.$(F;\mathcal{W})$ but is not stable $(F;\mathcal{W})$. The set $A_2 = \{(0,t) : t \geqq 0\}$ is a.s.$(F;\mathcal{W})$. The set $A_3 = \{(1,t) : t \in R\}$ is neither stable $(F;\mathcal{W})$ nor q.a.s.$(F;\mathcal{W})$.

We will now give a few properties of sets which are stable relative to a system of cones $\mathcal{C}(F)$ defined on $X \times R$

by F and the uniformity $\mathcal{W}$.

Lemma 3.16. Let $G:R \to 2^{X\times R}$ such that $G(t) \neq \emptyset$ if $t \geqq t_o$ for some $t_o \in R$ and let $A \subset X \times R$ be non-empty. $G(t) \to A$ as $t \to \infty$ if and only if $G(t) \to \bar{A}$ as $t \to \infty$. (Closure in the uniform topology of $X \times R$.)

Proof: We will carry out this proof in explicit detail in order to show how the lemmata of section C Chapter 2 are applied. In the theorems which follow this lemma, the properties which the lemmata of section C Chapter 2 will be used without mention.

Suppose $G(t) \to A$ as $t \to \infty$ then by definition, given $W \in \mathcal{W}$ there exists a $T(W) \in R$ such that $G(t) \subset W[A]$ whenever $t \geqq t_o + T(W)$. By Lemma 2.22, $W[A] \subset W[\bar{A}]$ since $A \subset \bar{A}$. Hence $G(t) \subset W[\bar{A}]$ whenever $t \geqq t_o + T(W)$ and by definition $G(t) \to \bar{A}$ as $t \to \infty$.

Suppose that $G(t) \to \bar{A}$ as $t \to \infty$ and let $W \in \mathcal{W}$ be given. Choose $W_1 \in \mathcal{W}$ such that $W_1 \circ W_1 \subset W$. Corresponding to W_1 there is a $T(W_1) \in R$ such that $G(t) \subset W_1[\bar{A}]$ whenever $t > t_o + T(W_1)$ by definition. By Lemma 2.24 $\bar{A} \subset W_1[A]$ and by Lemma 2.22, $W_1[\bar{A}] \subset W_1[W_1[A]]$, so that $W_1[\bar{A}] \subset W_1 \circ W_1[A]$ by Lemma 2.23. Since $W_1 \circ W_1 \subset W$ we have that $W_1[\bar{A}] \subset W[A]$ by Lemma 2.22. Hence $G(t) \subset W[A]$

whenever $t > t_o + T(W_1)$ and we conclude $G(t) \to A$ as $t \to \infty$.

This lemma finds application in the following:

Proposition 3.17. If $\bar{A}$ is q.a.s.$(F;\mathcal{W})$ then A is q.a.s.$(F;\mathcal{W})$. If A is u.q.a.s.$(F;\mathcal{W})$ then $\bar{A}$ is u.q.a.s.$(F;\mathcal{W})$.

Proof: If $\bar{A}$ is q.a.s.$(F;\mathcal{W})$ then given $p_o \in \bar{A}$ there is a $V_{p_o} \in \mathcal{W}$ such that $F_t(p_1) \to \bar{A}$ as $t \to \infty$ for all $p_1 \in V_{p_o}[p_o]$. Thus since $A \subset \bar{A}$, this property holds for all $p_o \in A$. By Lemma 3.16 $F_t(p_1) \to \bar{A}$ as $t \to \infty$ implies that $F_t(p_1) \to A$ as $t \to \infty$. Hence A is q.a.s.$(F;\mathcal{W})$.

Suppose that A is u.q.a.s.$(F;\mathcal{W})$. There exists a $V \in \mathcal{W}$ such that $F_t(p_o) \to A$ as $t \to \infty$ whenever $p_o \in V[A]$. Choose $V_1 \in \mathcal{W}$ such that $V_1 \circ V_1 \subset V$. If $p_o \in V_1[\bar{A}] \subset V_1 \circ V_1[A] \subset V[A]$ then $F_t(p_o) \to A$ as $t \to \infty$ and by Lemma 3.16 $F_t(p_o) \to \bar{A}$ as $t \to \infty$. Hence $\bar{A}$ is u.q.a.s.$(F;\mathcal{W})$.

Proposition 3.18. A non-empty subset A of $X \times R$ is stable $(F;\mathcal{W})$ if and only if $\bar{A}$ is stable $(F;\mathcal{W})$.

Proof: See Bushaw [3]. The proof therein is applied by using Remark 3.8.

Corollary 3.19. Let $A \subset X \times R$ be non-empty. If $\bar{A}$ is a.s.$(F;\mathcal{W})$ then A is a.s.$(F;\mathcal{W})$. If A is u.a.s.$(F;\mathcal{W})$ then $\bar{A}$ is u.a.s.$(F;\mathcal{W})$.

Proof: This follows from Definitions 3.11 and 3.12 and Propositions 3.17 and 3.18.

Remark 3.20. In Example 3.15, let $A = \{(x,t) : t > 0, -e^{-t} < x < e^{-t}\}$. A is q.a.s.$(F;\mathcal{W})$. However, A is not u.q.a.s.$(F;\mathcal{W})$, and it is clear that $\bar{A} = \{(x,t) : t \geqq 0, -e^{-t} \leqq x \leqq e^{t}\}$ is not q.a.s.$(F;\mathcal{W})$. Also it can be noted that A is not stable $(F;\mathcal{W})$. This shows that it is not possible to remove the assumption that A is u.q.a.s. which appears in Proposition 3.17 and Corollary 3.19 cannot be weakened and thus within the framework of our definitions this is the best possible result.

Definition 3.21. A non-empty subset A of $X \times R$ will be called invariant with respect to the system of cones $\mathcal{C}(F)$ defined on $X \times R$ by F, or more simply F-invariant, in case $F(p_o) \subset A$ for all $p_o \in A$.

Stability concepts are usually defined for sets which are invariant in some sense. The fact that no hypothesis of invariance is made is, perhaps, the most striking feature of

Definition 3.10. We will show in the following proposition that under our definition of stability, certain stable sets are invariant.

Proposition 3.22. A closed, non-empty stable $(F;\mathcal{W})$ subset of $X \times R$ is F-invariant.

Proof: Again the proof is due to Bushaw [3] and can be applied here by using Remark 3.8.

C. Necessary and Sufficient Conditions for Stability $(F;\mathcal{W})$ and Asymptotic Stability $(F;\mathcal{W})$

We will now give necessary and sufficient conditions for a non-empty subset A of $X \times R$ to be stable with respect to a system of cones $\mathcal{C}(F)$ and uniformity $\mathcal{W}$. We will also give necessary and sufficient conditions in order that a nonempty subset A of $X \times R$ be quasi-asymptotically stable with respect to a system of cones $\mathcal{C}(F)$ and uniformity $\mathcal{W}$. Our conditions will be given in terms of properties of a certain function called a Lyapunov function with infinitesimal upper bound. The following definition is an elaboration on the definition given by Bushaw [3] which was, in turn, a generalization of the classical real valued Lyapunov function. (See [15] or [1])

Let $(M,\leqq)$ be a partially ordered set which has a least element 0 (that is, $m \geqq 0$ for all $m \in M$) and such that $M\setminus\{0\}$ is not empty. Let $\mathcal{B}(F)$ be a system of cones defined on $X \times R$ by F and let A be a non-empty subset of $X \times R$. Let N be an F-invariant uniform neighborhood of A (that is, there exists a $V \in \mathcal{W}$ such that $V[A] \subset N$ and $F(p_o) \subset N$ for all $p_o \in N$).

Definition 3.23. A mapping $L:N \times R \to M$ is called a Lyapunov function with infinitesimal upper bound for A if it satisfies:

L_1) If $p_o \in N$, $p_1 \in F(p_o)$, $s,t \in R$ with $s \geqq t$ then $L(p_1,s) \leqq L(p_o,t)$.

L_2) For any given $W \in \mathcal{W}$, there is a $\lambda \in M\setminus\{0\}$ such that if $p_o \in N\setminus W[A]$ then $L(p_o,t_o) \geqq \lambda$.

L_3) For any $\lambda \in M\setminus\{0\}$ given, there is a $W \in \mathcal{W}$ such that $L(p_o,t_o) \ngeqq \lambda$ if $p_o \in W[A] \cap N$.

L_4) There is an open set 0_A containing A such that if $p_1 \in 0_A$ and $\lambda \in M\setminus\{0\}$ are given then there exists a $T(\lambda,p_1) \in R$ such that $L(p_1,t) \ngeqq \lambda$ whenever $t > t_1 + T(\lambda,p_1)$.

If L_1), L_2) and L_3) are satisfied by $L:N \times R \to M$ then L will be called a Lyapunov function for A.

Let us examine the Lyapunov function defined above and attempt to bring to light some of its properties and discover analogies between this function and the classical Lyapunov function.

<u>Proposition 3.24</u>. If $L: N \times R \to M$ is a Lyapunov function for $A \subset X \times R$ then $L(p_o, t_o) = 0$ if and only if $p_o \in \bar{A}$.

Proof: Suppose $L(p_o, t_o) = 0$ and suppose that for some $W \in \mathcal{W}$ we have that $p_o \in N \backslash W[A]$. By L_2) there is a $\lambda \in M\backslash\{0\}$ such that $L(p_o, t_o) \geqq \lambda \neq 0$, a contradiction. Hence $p_o \in W[A]$ for all $W \in \mathcal{W}$ so that $p_o \in \cap \{W[A] : W \in \mathcal{W}\} = \bar{A}$ by Lemma 2.24.

Conversely, if $p_o \in \bar{A} = \cap \{W[A] : W \in \mathcal{W}\}$ then for any $W \in \mathcal{W}$ $p_o \in W[A]$. Suppose that $L(p_o, t_o) = \alpha \neq 0$, that is $\alpha \in M\backslash\{0\}$. By L_3), there is a $W_1 \in \mathcal{W}$ such that if $p_1 \in W_1[A] \cap N$ then $L(p_1, t_1) \ngeqq \alpha$. In particular we have seen that $p_o \in W_1[A]$ and since $p_o \in U[A]$ for all $U \in \mathcal{W}$; $p_o \in N$ since N is a uniform neighborhood of A. Thus $p_o \in W_1[A] \cap N$ so that $L(p_o, t_o) \ngeqq \alpha$ which establishes a contradiction. Hence $L(p_o, t_o) = 0$ if $p_o \in \bar{A}$.

This property has its analogue in the fact that the classical Lyapunov function is usually taken to be positive definite with respect to the stable set.

There are other analogies which are worth mentioning. The most obvious one is the interpretation of L_1) as saying that $L(p,t)$ decreases jointly with respect to the quasi-order defined on $X \times R$ by the system of cones $\mathcal{C}(F)$ and the usual order on R. This is much like the requirement that the classical Lyapunov function is non-increasing along solution trajectories which begin "close" enough to the null solution. Condition L_2) says that $L(p_o,t_o)$ is bounded away from zero outside uniform neighborhoods of the set A and condition L_3) says that $L(p_o,t_o)$ approaches 0 uniformly as $p_o = (x_o,t_o)$ approaches A. These two conditions are analogous to properties of the classical Lyapunov function which are usually derived in proofs of stability theorems. Finally, our condition L_4) says that for each p_o belonging to some open set, $L(p_o,t)$ approaches 0 for sufficiently large t. We note that this condition is not uniform in p. This differs from its classical counterpart which is the concept of the infinitesimal upper bound given in definition 1.5.

In what follows, we take the partially ordered set M to be the retracted scale $\mathcal{P}_r(\mathcal{W})$ of $\mathcal{W}$. The selection of this set as the range of our Lyapunov functions will allow us to obtain necessary and sufficient conditions for a non-empty set $A \subset X \times R$ to the stable $(F;\mathcal{W})$ and a.s.$(F;\mathcal{W})$ where $F:X \times R \to 2^{X\times R}$ is given and defines a system of cones on $X \times R$.

<u>Theorem 3.25</u>. Suppose $\mathcal{B}(F)$ is a system of cones on $X \times R$ and that $A \subset X \times R$ is non-empty. A is stable $(F;\mathcal{W})$ if and only if there exists a Lyapunov function for A into $\mathcal{P}_r(\mathcal{W})$.

Proof: This result is due to Bushaw [3] and the proof will not be repeated here.

We now give a necessary and sufficient condition for a non-empty subset A of $X \times R$ to be quasi-asymptotically stable with respect to a system of cones $\mathcal{B}(F)$ and the uniformity $\mathcal{W}$.

<u>Theorem 3.26</u>. Suppose $\mathcal{B}(F)$ is a system of cones on $X \times R$ and that $A \subset X \times R$ is non-empty. A is q.a.s.$(F;\mathcal{W})$ if and only if there exists a mapping $L:N \times R \to \mathcal{P}_r(\mathcal{W})$ where N is an F-invariant uniform neighborhood of A, which satisfies L_1), L_2) and L_4).

Proof: We first show that the condition is sufficient. Suppose that $L:N \times R \to \mathcal{P}_r(\mathcal{W})$ satisfies L_1), L_2) and L_4)

with N an F-invariant uniform neighborhood of A. Since N is a uniform neighborhood of A there is a $V_N \in \mathcal{W}$ such that $V_N[A] \subset N$. By L_4) there is an open set O_A containing A and hence for any $p_o \in A$ there is a $V_{p_o} \in \mathcal{W}$ such that $V_{p_o}[p_o] \subset O_A$. Let $U_{p_o} = V_N \cap V_{p_o}$ then $U_{p_o} \in \mathcal{W}$ and $U_{p_o}[p_o] \subset N \cap O_A$. We claim that for all $p_1 \in U_{p_o}[p_o]$ we have $F_t(p_1) \to A$ as $t \to \infty$. In order to show that this is indeed the case we assume the contrary; that is, for some $p_1 \in U_{p_o}[p_o]$, $F_t(p_1)$ does not approach A as $t \to \infty$. Thus for some $W \in \mathcal{W}$, $F_t(p_1) \not\subset W[A]$ for all $t \in R$. This is the case since $F_\tau(p_1) \subset W[A]$ for some $\tau \in R$ implies that $F_t(p_1) \subset W[A]$ for all $t \geqq \tau$ because $F_t(p_1) \subset F_\tau(p_1)$ for $t \geqq \tau$. Since N is F-invariant $F(p_1) \subset N$ so that we have $F_t(p_1) \cap (N \backslash W[A]) \neq \emptyset$ for all $t \in R$.

We now select a sequence of points in $F(p_1)$. Choose $\hat{p}_1 = (\hat{x}_1, \hat{t}_1) \in F(p_1) \cap (N \backslash W[A])$ with $\hat{t}_1 \geqq 1$. Choose $\hat{p}_2 = (\hat{x}_2, \hat{t}_2) \in F_{\hat{t}_1}(p_1) \cap (N \backslash W[A])$ such that $\hat{t}_2 \geqq 2\hat{t}_1$. Inductively, choose $\hat{p}_{k+1} = (\hat{x}_{k+1}, \hat{t}_{k+1}) \in F_{\hat{t}_k}(p_1) \cap (N \backslash W[A])$ such that $\hat{t}_{k+1} \geqq (k+1)\hat{t}_k$ for $k = 1,2,3\cdots$. We note that the sequence $\{\hat{p}_k\}_{k=1}^{\infty} \subset F(p_1) \cap (N \backslash W[A])$ and that the sequence $\{\hat{t}_k\}_{k=1}^{\infty} \to \infty$ since $\hat{t}_k \geqq k!$.

By $L_2)$ there is a $\lambda \in \mathcal{P}_r(\mathcal{W})\backslash\{0\}$ such that if $p' = (x',t') \in N\backslash W[A]$ then $L(p',t') \geqq \lambda$. Hence, in particular, $L(\hat{p}_k,\hat{t}_k) \geqq \lambda$ for each $\hat{p}_k = (\hat{x}_k,\hat{t}_k)$ of the sequence we have just chosen.

Let $\alpha = \bigcup_{k=1}^{\infty} L(\hat{p}_k,\hat{t}_k)$. $\alpha \in \mathcal{P}(\mathcal{W})$ since $L(\hat{p}_k,\hat{t}_k) \in \mathcal{P}_r(\mathcal{W})$ for $k = 1,2,3\cdots$. Since $L(\hat{p}_k,\hat{t}_k) \geqq \lambda$ we have $L(\hat{p}_k,\hat{t}_k) \subseteqq \lambda$ for $k = 1,2,3\cdots$, and hence $\alpha \subseteqq \lambda$ which in turn implies that $\lambda \leqq \alpha$. If we let $\beta = w(\alpha)$ where w is the mapping defined in 2-1, then since w is order preserving and $w(\gamma) = \gamma$ for all $\gamma \in \mathcal{P}_r(\mathcal{W})$ we have $\lambda = w(\lambda) \leqq w(\alpha) = \beta$. Thus $\beta \in \mathcal{P}_r(\mathcal{W})\backslash\{0\}$.

By $L_4)$, using β and p_1 we have that there is a $T(\beta,p_1) \in R$ such that $L(p_1,t) \ngeqq \beta$ for all $t > t_1 + T(\beta,p_1)$. Since the sequence $\{\hat{t}_k\}_{k=1}^{\infty}$ diverges, there is a positive integer M such that $\hat{t}_k > t_1 + T(\beta,p_1) + 1$ if $k \geqq M$. Fix $m > M$. Since $\hat{p}_m \in F(p_1)$ we have by $L_1)$ that $L(\hat{p}_m,\hat{t}_m) \leqq L(p_1,t)$ for all $t \leqq \hat{t}_m$. Thus, in particular, $L(\hat{p}_m,\hat{t}_m) \leqq L(p_1,t)$ for $t_1 + T(\beta,p_1) < t \leqq \hat{t}_m$.

Now by definition $L(\hat{p}_m,\hat{t}_m) \subset \alpha$ so that $\alpha \leqq L(\hat{p}_m,\hat{t}_m)$ and since $L(\hat{p}_m,\hat{t}_m) \in \mathcal{P}_r(\mathcal{W})$ and w is order preserving we have $\beta = w(\alpha) \leqq w(L(\hat{p}_m,\hat{t}_m)) = L(\hat{p}_m,\hat{t}_m)$. Hence, by transitivity of the partial ordering $\leqq$, $\beta \leqq L(p_1,t)$ for $t_1 + T(\beta,p_1) < t \leqq \hat{t}_m$ which contradicts the choice of $T(\beta,p_1)$. Therefore, for each $W \in \mathcal{W}$, $F_\tau(p_1) \subset W[A]$ for some $\tau \in R$ (and consequently $F_t(p_1) \subset W[A]$ for all $t \geqq \tau$) and for all $p_1 \in U_{p_o}[p_o]$. Thus $F_t(p) \to A$ as $t \to \infty$ for all $p \in U_{p_o}[p_o]$ and since $p_o \in A$ is arbitrary, A is q.a.s.$(F;\mathcal{W})$.

Let us now suppose that A is a non-empty subset of $X \times R$ which is q.a.s.$(F;\mathcal{W})$. We define the mapping $\Gamma(A,F;\cdot,\cdot):(X \times R) \times R \to \mathcal{P}(\mathcal{W})$ by:

$$(3\text{-}6) \qquad \Gamma(A,F;p_o,t) = \{W \in \mathcal{W} : F_t(p_o) \subset W[A]\}$$

for each $p_o = (x_o,t_o) \in X \times R$ and $t \in R$. It is clear that $\Gamma(A,F;p_o,t) \in \mathcal{P}(\mathcal{W})$ for each $p_o \in X \times R$ and $t \in R$ since $(X \times R)\times(X \times R) \in \Gamma(A,F;p_o,t)$ for each $p_o \in X \times R$ and $t \in R$, and if $U_1 \in \Gamma(A,F;p_o,t)$ and $U_2 \in \mathcal{W}$ such that $U_1 \subset U_2$ then $U_2 \in \Gamma(A,F;p_o,t)$ since $F_t(p_o) \subset U_2[A]$. We note that $\Gamma(A,F;p_o,t)$ also depends on the subset A of $X \times R$ and system of cones $\mathcal{G}(F)$. However, this consideration does not enter at this point and discussion will be deferred.

We now define the mapping $L(A,F;\cdot,\cdot):(X \times R) \times R \to \mathcal{P}_r(\mathcal{W})$ by:

$$(3\text{-}7) \qquad L(A,F;\cdot,\cdot) = w \circ \Gamma(A,F;\cdot,\cdot)$$

where $\circ$ denotes composition and $w:\mathcal{P}(\mathcal{W}) \to \mathcal{P}_r(\mathcal{W})$ is the mapping defined by (2-1). We will show that $L(A,F;\cdot,\cdot)$ satisfies L_1) and L_2) for any arbitrary non-empty $A \subset X \times R$ and that condition L_4) is satisfied when A is q.a.s.$(F;\mathcal{W})$.

Let $p_o \in X \times R$ be given and let $p_1 \in F(p_o)$, then by C_3) of Definition 3.2, $F(p_1) \subset F(p_o)$ so that for any $t \in R$, $F_t(p_1) \subset F_t(p_o)$. From Definition 3.7 it follows that if $s, t \in R$ with $s \geqq t$ then $F_s(p_1) \subset F_t(p_1)$. Hence, combining these facts, we have $F_s(p_1) \subset F_t(p_o)$ whenever $s \geqq t$ and $p_1 \in F(p_o)$.

If $W \in \Gamma(A,F;p_o,t)$ then $F_t(p_o) \subset W[A]$ so that if $p_1 \in F(p_o)$ and $s, t \in R$ with $s \geqq t$ we have that $F_s(p_1) \subset F_t(p_o) \subset W[A]$. Thus $W \in \Gamma(A,F;p_1,s)$, and it follows that $\Gamma(A,F;p_o,t) \subset \Gamma(A,F;p_1,s)$. In the ordering we have imposed on $\mathcal{P}(\mathcal{W})$ this means that

$\Gamma(A,F;p_1,s) \leqq \Gamma(A,F;p_o,t)$ and since w is order preserving we obtain $L(A,F;p_1,s) = w(\Gamma(A,F;p_1,s)) \leqq w(\Gamma(A,F;p_o,t)) = L(A,F;p_o,t)$ which is precisely condition L_1).

Let $W \in \mathcal{W}$ be given. We note that the uniform neighborhood of A upon which $L(A,F;\cdot,\cdot)$ is defined is, in this case, the whole space $X \times R$. We may assume that $(X \times R)\backslash W[A] \neq \emptyset$ for if $W[A] = X \times R$ then L_2) holds vacuously. Choose $W_1 \in \mathcal{W}$ such that $W_1 \circ W_1 \subset W$. Let $\alpha = \cup\{L(A,F;p_o,t_o)\colon p_o \notin W[A]\}$. $\alpha \in \mathcal{P}(\mathcal{W})$ since it is the union of members of $\mathcal{P}_r(\mathcal{W})$ and we claim that $\alpha \neq \mathcal{W}$. If $\alpha = \mathcal{W}$ then $W_1 \in \alpha$ so that for some $p_1 = (x_1,t_1) \notin W[A]$, $W_1 \in L(A,F;p_1,t_1)$. Now, be definition:

$$\begin{aligned} L(A,F;p_1,t_1) &= w(\Gamma(A,F;p_1,t_1)) \\ &= \{V \in \mathcal{W}\colon F_{t_1}(p_1) \subset Z \circ V[A] \text{ for all } Z \in \mathcal{W}\}. \end{aligned}$$

Hence $F_{t_1}(p_1) \subset Z \circ W_1[A]$ for all $Z \in \mathcal{W}$ and, in particular, $F_{t_1}(p_1) \subset W_1 \circ W_1[A]$ which in turn implies $F_{t_1}(p_1) \subset W[A]$. But $p_1 \in F_{t_1}(p_1)$ since $F_{t_1}(p_1) = F(p_1)$ and so $p_1 \in W[A]$ which is a contradiction. Therefore $W_1 \notin \alpha$ and $\alpha \neq \mathcal{W}$.

We now let $\beta = w(\alpha)$. $\beta \neq \mathcal{W}$ since $\alpha \neq \mathcal{W}$ so that $\beta \in \mathcal{P}_r(\mathcal{W})\backslash\{0\}$. For any $p_o = (x_o,t_o) \in (X \times R)\backslash W[A]$ we have $L(A,F;p_o,t_o) \subset \alpha$ so that $L(A,F;p_o,t_o) \geqq \alpha$ and by the order preserving property of w and the fact that $L(A,F;p_o,t_o)$ is a member of $\mathcal{P}_r(\mathcal{W})$ we have $L(A,F;p_o,t_o) \geqq \beta$ which is property L_2).

Since A is q.a.s.$(F,\mathcal{W})$, for each $p_o \in A$ there is a $V_{p_o} \in \mathcal{W}$ such that $F_t(p_1) \to A$ as $t \to \infty$ for all $p_1 \in V_{p_o}[p_o]$. Choose such a $V_{p_o} \in \mathcal{W}$ for each $p_o \in A$ and let $S_A = \cup\{V_{p_o}[p_o] : p_o \in A\}$ and

$O_A = \text{int } S_A = \{p_1 \in S_A :$ there is $U \in \mathcal{W}$ such that $U[p_1] \subset S_A\}$. O_A is certainly an open set containing A. Let $p_1 \in O_A$ then $p_1 \in V_{p_o}[p_o]$ for some $p_o \in A$. Let $\lambda \in \mathcal{P}_r(\mathcal{W})\backslash\{0\}$ be given and choose $\overline{W} \in \mathcal{W}\backslash\lambda$. Since A is q.a.s.$(F,\mathcal{W})$ and $p_1 \in V_{p_o}[p_o]$, $F_t(p_1) \to A$ as $t \to \infty$ so that there is a $T(W,p_1) \in R$ such that $F_t(p_1) \subset W[A]$ if $t > t_1 + T(W,p_1)$.

If $L(A,F;p_1,\tau) \ngeqq \lambda$ for some $\tau \in R$ then $L(A,F;p_1,s) \ngeqq \lambda$ for all $s \geqq \tau$ because L_1) holds. Thus we suppose that $L(A,F;p_1,t) \geqq \lambda$ for all $t \in R$, or what is the same, $L(A,F;p_1,t) \subset \lambda$ for all $t \in R$. Now

$W \in \Gamma(A,F;p_1,t)$ for all $t > t_1 + T(W,p_1)$ so that $W \in L(A,F;p_1,t)$ for all $t > t_1 + T(W,p_1)$ since $\Gamma(A,F;p,\tau) \subset L(A,F,p,\tau)$ for all $p \in X \times R$ and $\tau \in R$. But this means that $W \in \lambda$ which contradicts the fact that $W \in \mathcal{W}\backslash\lambda$. Therefore, there is a $t_2 \in R$ such that $L(A,F;p_1,t_2) \not\geqq \lambda$ and hence as we noted before $L(A,F;p_1,t) \not\geqq \lambda$ for $t \geqq t_2$. Setting $T(\lambda,p_1) = \max(0,t_2-t_1)$ we have $L(A,F;p_1,t) \not\geqq \lambda$ for $t \geqq t_1 + T(\lambda,p_1)$ which exactly L_4).

Corollary 3.27. Suppose $\mathcal{C}(F)$ is a system of cones on $X \times R$ and that A is a non-empty subset of $X \times R$. A is a.s.$(F,\mathcal{W})$ if and only if there exists a Lyapunov function with infinitesimal upper bound for A into the retracted scale of $\mathcal{W}$.

Proof: This follows immediately from Theorems 3.25 and 3.26 and Definition 3.12.

IV. CONTINUITY OF THE MAPPING $L(A,F;p,\tau)$

In this chapter we will investigate the circumstances under which the mapping $L(A,F;\cdot,\cdot):(X \times R) \times R \to \mathcal{P}_r(\mathcal{W})$, defined in equations (3-6) and (3-7), is continuous. We note that this mapping is well-defined for any non-empty subset A of $X \times R$ and system of cones $\mathcal{B}(F)$ on $X \times R$ and that, indeed, conditions L_1) and L_2) of Definition 3.23 are always satisfied. Bushaw [3] has shown that the mapping satisfies L_3) in case A is stable $(F;\mathcal{W})$ and we have shown in Theorem 3.26 that it satisfies L_4) in case A is q.a.s.$(F;\mathcal{W})$. That $L(A,F;\cdot,\cdot)$ possess these latter properties will not be necessary to our arguments, and hence, we prefer to allow A to be an arbitrary non-empty subset of $X \times R$ and $\mathcal{B}(F)$ any system of cones on $X \times R$. Later in the chapter we will make stronger assumptions concerning the form and stability properties of A and show that these considerations have a marked effect on the continuity of $L(A,F;\cdot,\cdot)$.

The problem we have set ourselves actually reduces to that of studying continuity of the mapping $\Gamma(A,F;\cdot,\cdot)$ defined in (3-6). This follows since Proposition 2.21

states that $w: \mathcal{P}(\mathcal{W}) \to \mathcal{P}_r(\mathcal{W})$ is uniformly continuous relative to the uniformities $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}_r$ and $L(A,F;\cdot,\cdot)$ is the usual composition of the mappings w and $\Gamma(A,F;\cdot,\cdot)$.

The topologies which we deal with in studying continuity of $\Gamma(A,F;\cdot,\cdot)$ are those induced on $X \times R \times R$ and on $\mathcal{P}(\mathcal{W})$ by the uniformities $\mathcal{Z}$ and $\tilde{\mathcal{W}}$ respectively. Our mode of attack for showing that $\Gamma(A,F;\cdot,\cdot)$ is continuous at a point $(p_o,\tau) \in (X \times R) \times R$ will be to show that the inverse image of a neighborhood of $\Gamma(A,F;p_o,\tau)$ in $\mathcal{P}(\mathcal{W})$ is a neighborhood of (p_o,τ) in $(X \times R) \times R$ (see Kelley [5]). Since sets of the form $\tilde{W}[\Gamma(A,F;p_o,\tau)]$ where $W \in \mathcal{W}$ form a base for a neighborhood system at $\Gamma(A,F;p_o,\tau) \in \mathcal{P}(\mathcal{W})$ and sets of the form $Z[(p_o,\tau)]$ where $Z \in \mathcal{Z}$ form a base for a neighborhood system at $(p_o,\tau) \in (X \times R) \times R$, it suffices to show that given $W \in \mathcal{W}$ there is a $Z \in \mathcal{Z}$ such that $\Gamma(A,F;p_o',\tau') \in \tilde{W}[\Gamma(A,F;p_o,\tau)]$ whenever $(p_o',\tau') \in Z[(p_o,\tau)]$ in order to conclude that $\Gamma(A,F;\cdot,\cdot)$ is continuous at (p_o,τ).

Let us see what conditions must be fulfilled in order that $\Gamma(A,F;\cdot,\cdot)$ is continuous at (p_o,τ). Let W be given and consider the set:

$$\widetilde{W}[\Gamma(A,F;p_o,\tau)] = \{\alpha \in \mathcal{P}(\mathcal{W}) : (\Gamma(A,F;p_o,\tau),\alpha) \in \widetilde{W}\}$$

In order that $\Gamma(A,F;p_1,s) \in \widetilde{W}[\Gamma(A,F;p_o,\tau)]$ for some $(p_1,s) \in (X \times R) \times R$ we must have $(\Gamma(A,F;p_o,\tau),\ \Gamma(A,F;p_1,s)) \in \widetilde{W}$, which in turn is equivalent to: $W\langle\Gamma(A,F;p_o,\tau)\rangle \subset \Gamma(A,F;p_1,s)$ and

$W\langle\Gamma(A,F;p_1,s)\rangle \subset \Gamma(A,F,p_o,\tau)$.

We may rewrite this statement in an equivalent form by using the definition of $\Gamma(A,F;\cdot,\cdot)$.

$(\Gamma(A,F;p_o,\tau),\ \Gamma(A,F;p_1,s)) \in \widetilde{W}$ if and only if both

(4-1) for any $U \in \mathcal{W}$ such that $F_\tau(p_o) \subset U[A]$ it follows that $F_s(p_1) \subset W \circ U[A]$

(4-2) for any $V \in \mathcal{W}$ such that $F_s(p_1) \subset V[A]$ it follows that $F_\tau(p_o) \subset W \circ V[A]$.

hold.

It is clear now that our continuity arguments reduce to showing that given $W \in \mathcal{W}$, there is a $Z \in \mathcal{Z}$ such that (4-1) and (4-2) hold whenever $(p_1,s) \in Z[(p_o,\tau)]$.

Obviously, we will now need to make some further assumptions concerning $\mathscr{C}(F)$ and A. Actually, we are offered two courses of action. We may place restrictions on A and leave $\mathscr{C}(F)$ alone or we may impose more conditions on $\mathscr{C}(F)$ and let A be arbitrary. The latter course seems the most tempting at the outset since systems of cones which arise in applications usually possess many more properties than we have had occasion to use thus far. However, examination of conditions (4-1) and (4-2) shows that continuity of $L(A,F;\cdot,\cdot)$ at a point (p_o,τ) is a matter of having the "greatest distance" of $F_s(p_1)$ from A be a continuous function of (p,t) at (p_o,τ). Thus it is that the behaviour of the system of cones relative to the given set A is of the essence rather than the behaviour of the individual cones of the system relative to one another. This observation shows us that if we are interested in having continuity of $L(A,F;\cdot,\cdot)$ for an arbitrary non-empty set A, we will have to impose very stringent conditions on the system of cones. Hence, it is the case that we can achieve the most satisfactory results by imposing certain restrictions on the sets A which we will consider.

<u>Proposition 4.1</u>. Suppose $\mathscr{C}(F)$ is a system of cones on $X \times R$ and A is a non-empty subset of $X \times R$ which is

stable $(F;\mathcal{W})$ then $L(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$.

Proof: Let $W \in \mathcal{W}$ and $(p_o,\tau) \in \bar{A} \times R$ be given. Choose $W_1 \in \mathcal{W}$ such that $W_1 \circ W_1 \subset W$. Since A is stable $(F;\mathcal{W})$, $\bar{A}$ is stable $(F;\mathcal{W})$ by Proposition 3.18, and hence, there is a $W_2 \in \mathcal{W}$ such that $F(p') \subset W_1[\bar{A}]$ whenever $p' \in W_2[\bar{A}]$. Suppose $U \in \mathcal{W}$ is such that $F_\tau(p_o) \subset U[A]$. For any $p_1 \in W_2[p_o] \subset W_2[\bar{A}]$ we have that $F(p_1) \subset W_1[\bar{A}]$, and since $\bar{A} \subset W_1[A]$ and $W_1 \circ W_1 \subset W$, $W_1[\bar{A}] \subset W_1[W_1[A]] \subset W[A]$. Hence for $p_1 \in W_2[p_o]$ and any $s \in R$, $F_s(p_1) \subset F(p_1) \subset W[A] \subset W[U[A]] = W \circ U[A]$ and (4-1) holds provided only that $p_1 \in W_2[p_o]$.

Now since $\bar{A}$ is closed and stable $(F;\mathcal{W})$, $\bar{A}$ is F-invariant by Proposition 3.22, so that $p_o \in \bar{A}$ implies $F_\tau(p_o) \subset \bar{A} \subset U[A]$ for all $U \in \mathcal{W}$. Hence for any $p_1 \in X \times R$ and $s \in R$, if $U \in \mathcal{W}$ is such that $F_s(p_1) \subset U[A]$ then $F_\tau(p_o) \subset W \circ U[A]$ and (4-2) holds for arbitrary $(p_1,s) \in (X \times R) \times R$.

Choose $U_2 \in \mathcal{U}$, $V_2 \in \mathcal{R}$ such that $S(U_2,V_2) \subset W_2$ and let $Z = S(U_2,V_2,R \times R)$. If $(p_1,s) \in Z[(p_o,\tau)]$ then $p_1 \in W_2[p_o]$ so (4-1) and (4-2) hold. Hence $\Gamma(A,F;\cdot,\cdot)$ is continuous at each point $(p_o, \tau) \in \bar{A} \times R$. We now note that the choice of Z is independent of the particular point $(p_o, \tau) \in \bar{A} \times R$. Therefore, $\Gamma(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$ and since w is uniformly continuous relative to the uniformities $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}_r$, $L(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$.

If we impose a further restriction on A it is possible to obtain a converse for Proposition 4.1.

<u>Proposition 4.2</u>. Suppose that $\mathcal{C}(F)$ is a system of cones on $X \times R$ defined by F and that $A \subset X \times R$ is such that $\bar{A}$ is F-invariant. If $\Gamma(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$ then A is stable $(F;\mathcal{W})$.

Proof: Let $W \in \mathcal{W}$ be given. Choose $W_1 \in \mathcal{W}$ such that $W_1 \circ W_1 \subset W$. By the uniform continuity of $\Gamma(A,F;\cdot,\cdot)$ on $\bar{A} \times R$, there is a $Z_1 \in \mathcal{Z}$ such that $(\Gamma(A,F;p_o,\tau), \Gamma(A,F;p_1,s)) \in \tilde{W}_1$ whenever $((p_o,\tau),(p_1,s)) \in Z_1$ for any $(p_o,\tau) \in \bar{A} \times R$. Thus, in particular, we have that $W_1\langle\Gamma(A,F;p_o,\tau)\rangle \subset \Gamma(A,F;p_1,s)$ whenever $((p_o,\tau),(p_1,s)) \in Z_1$ for any $(p_o,\tau) \in \bar{A} \times R$. Using the definitions, we then have that if $U \in \Gamma(A,F;p_o,\tau)$ then $W_1 \circ U \in \Gamma(A,F;p_1,s)$ whenever $((p_o,\tau),(p_1,s)) \in Z_1$, or equivalently, if $U \in \mathcal{W}$ such that $F_\tau(p_o) \in U[A]$ then $F_s(p_1) \in W_1 \circ U[A]$ provided only that $((p_o,\tau),(p_1,s)) \in Z_1$.

Choose $U_2 \in \mathcal{U}$, $V_2 \in \mathcal{R}$ such that $S(U_2,V_2,V_2) \subset Z_1$. Let $W_2 = S(U_2,V_2)$. If $p_1 \in W_2[p_o]$ then $t_1 \in V_2[t_o]$ and thus $((p_o,t_o),(p_1,t_1)) \in S(U_2,V_2,V_2) \subset Z_1$

Let $p_o \in A$ then since $\bar{A}$ is F-invariant, $F_{t_o}(p_o) = F(p_o) \subset \bar{A} \subset U[A]$ for all $U \in \mathcal{W}$. It then follows that $F_{t_1}(p_1) = F(p_1) \subset W_1 \circ U[A]$ provided only that $p_1 \in W_2[p_o]$. In particular with $U = W_1$, we have that $F(p_1) \subset W_1 \circ W_1[A] \subset W[A]$ whenever $p_1 \in W_2[p_o]$. However, $p_o \in A$ is arbitrary so we have that $F(p_1) \subset W[A]$ whenever $p_1 \in W_2[A]$ and hence A is stable $(F,\mathcal{W})$.

There are two results which now follow immediately.

Corollary 4.3. Suppose $\mathcal{B}(F)$ is a system of cones defined on $X \times R$ and A is non-empty subset of $X \times R$. If $L(A,F;\cdot,\cdot)$ is a Lyapunov function for A then $L(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$.

Proof: The proof is immediate from Theorem 3.25 and Proposition 4.1.

Corollary 4.4. Under the same hypotheses as Corollary 4.3, if $\Gamma(A,F;\cdot,\cdot)$ is uniformly continuous on $\bar{A} \times R$ and $\bar{A}$ is F-invariant then $L(A,F;\cdot,\cdot)$ is a Lyapunov function for A.

Proof: The proof is immediate from Proposition 4.2 and the fact that $L(A,F;\cdot,\cdot)$ satisfies condition $L_3)$ when A is stable.

In conclusion we observe that if we ask only for continuity of $L(A,F;\cdot,\cdot)$ in τ for fixed p, we can answer affirmatively under rather general conditions on A and $\mathcal{B}(F)$. This is the case because continuity of $L(A,F;\cdot,\cdot)$ in τ for fixed $p \in X \times R$ depends only upon the behaviour of the cone $F(p)$. The following definition describes a more comprehensive property than we actually need. However, it is a property which is possessed by systems of cones to which we wish to apply our work, and, hence, we choose to introduce it here.

<u>Definition 4.5</u>. A cone $F(p_o)$ is said to be smooth at τ in case given $W \in \mathcal{W}$ there is a $\delta > 0$ such that $F_t(p_o) \subset W[F_\tau(p_o)]$ and $F_\tau(p_o) \subset W[F_t(p_o)]$ whenever $|t - \tau| < \delta$.

<u>Example 4.6</u>. It is very easy to give an example of a cone

which is not smooth. Let $X = R$ and for any $x_o, x_1 \in R$, $x_o \neq x_1$, t_o, $\tau \in R$, $\tau > t_o$ define $F(p_o) = \{(x_o,t) : t_o \leqq t < \tau\} \cup \{(x_1,t) : t \geqq \tau\}$. $F(p_o)$ is not smooth at τ. This follows since $F_\tau(p_o) = \{(x_1,t) : t \geqq \tau\}$ and we need only take $\epsilon = \dfrac{|x_1 - x_o|}{4}$ and $W = S(V_\epsilon, V_\epsilon)$. If $\delta = \epsilon/2$ we will have $F_\tau(p_o) \subset W[F_t(p_o)]$ whenever $|t - \tau| < \delta$ but $F_t(p_o) \not\subset W[F_\tau(p_o)]$ if $t < \tau$ so that <u>no</u> δ' can be found such that $F_t(p_o) \subset W[F_\tau(p_o)]$ for $|t - \tau| < \delta'$.

Using this concept we can prove the following result.

<u>Proposition 4.7</u>. Let A be a non-empty subset of $X \times R$ and $\mathcal{C}(F)$ be a system of cones defined on $X \times R$ by F. If $F(p_o)$ is smooth at τ then $L(A,F;p_o,\cdot)$ is continuous in t at $t = \tau$.

Proof: The argument, as usual, reduces to showing that given $W \in \mathcal{W}$ there is a $V \in \mathcal{R}$ such that (4-1) and (4-2) are satisfied when $t \in V[\tau]$. Suppose that $F(p_o)$ is smooth at τ and let $W \in \mathcal{W}$ be given. Choose $\delta > 0$ such that $F_t(p_o) \subset W[F_\tau(p_o)]$ and $F_\tau(p_o) \subset W[F_t(p_o)]$ whenever $|t - \tau| < \delta$. Set $V = V_\delta \in \mathcal{R}$. Let $U \in \mathcal{W}$ be such that

$F_\tau(p_o) \subset U[A]$ then $F_t(p_o) \subset W[F_\tau(p_o)] \subset W[U[A]] = W \circ U[A]$ whenever $t \in V[\tau]$ and thus (4-1) is satisfied for $t \in V[\tau]$. Similarly if we let $U \in \mathcal{W}$ be such that $F_t(p_o) \subset U[A]$ then $F_\tau(p_o) \subset W[F_t(p_o)] \subset W \circ U[A]$ whenever $t \in V[\tau]$ and thus (4-2) is satisfied for $t \in V[\tau]$. Hence $L(A,F;p_o,\cdot)$ is continuous in t at $t = \tau$.

V. APPLICATIONS TO GENERALIZED DYNAMICAL SYSTEMS

In this chapter we investigate applications of our asymptotic stability criteria to systems of cones generated by generalized dynamical systems (g.d.s.). We adapt the definition given by Roxin [9]. Let $(X,\mathcal{U})$ be an arbitrary uniform space.

<u>Definition 5.1</u>. A <u>generalized</u> <u>dynamical</u> <u>system</u> on X is given by a mapping $\psi: X \times R \times R \to 2^X$ which satisfies:

D_1) $\psi(x_o,t_o,t_o) = \{x_o\}$ for all $x_o \in X$, $t_o \in R$.

D_2) $\psi(x_o,t_o,t)$ is a non-empty closed subset of X for each $t \geqq t_o$.

D_3) For each $x_1 \in X$, t_o, $t_1 \in R$ with $t_o \leqq t_1$ there exists an $x_o \in X$ such that $x_1 \in \psi(x_o,t_o,t_1)$.

D_4) For $t_o \leqq t_1 \leqq t_2$, $\psi(x_o,t_o,t_2) = \cup \{\psi(x_1,t_1,t_2) : x_1 \in \psi(x_o,t_o,t_1)\}$.

D_5) Given $U \in \mathcal{U}$, $(x_o,t_o) \in X \times R$ and $\tau_o \geqq t_o$ there is a $\delta > 0$ (which, in general, depends on U, x_o, t_o, and τ_o) such that $\psi(x_o,t_o,\tau') \subset U[\psi(x_o,t_o,\tau_o)]$ and $\psi(x_o,t_o,\tau_o) \subset U[\psi(x_o,t_o,\tau')]$ whenever

$\tau' \in [\tau_o - \delta, \tau_o + \delta] \cap [t_o, \infty)$.

D_6) Given $U \in \mathcal{U}$, $(x_o, t_o) \in X \times R$ and any finite interval $[t_1, t_2]$ with $t_1 \geqq t_o$, there is a $V \in \mathcal{U}$ and a $\delta > 0$ (both of which, in general, depend on U, x_o, t_o, and the interval $[t_1, t_2]$) such that $\psi(x_o', t_o', t) \subset U[\psi(x_o, t_o, t)]$ for all $t \in [t_1, t_2] \cap [t_o^*, \infty)$, $t_o^* = \max(t_o, t_o')$, whenever $x_o' \in V[x_o]$ and $|t_o - t_o'| < \delta$.

A g.d.s. on X is sometimes referred to as a <u>generalized control system</u> (see Roxin [10] and [11]). It is common then to call the function ψ, the <u>attainability function</u> of the control system and interpret $\psi(x_o, t_o, t)$ as the set of points attainable from (x_o, t_o) at time t under the action of some set of controls.

In order to apply our results we must show that a g.d.s. defined on X by ψ generates a system of cones on $X \times R$. In order to do this we will let

$$\text{(5-1)} \qquad \Psi(p_o, t_1) = \{(x, t_1) : x \in \psi(x_o, t_o, t_1)\}$$

where $p_o = (x_o, t_o)$, $t_1 \geqq t_o$ and

(5-2) $\quad \hat{\Psi}(p_o) = \cup \{\Psi(p_o,t) : t \geqq t_o\}$.

Clearly for each $p_o \in X \times R$, $\hat{\Psi}(p_o) \in 2^{X\times R}$.

Lemma 5.2. $\hat{\Psi}(p_o)$ defined by (5-1) and (5-2) is a cone on p_o.

Proof: We need only show that C_1) and C_2) of Definition 3.2 are satisfied. From (5-1) and (5-2) we get:

$$s_{t_o}(\hat{\Psi}(p_o)) = s_{t_o}(\Psi(p_o,t_o)) = \psi(x_o,t_o,t_o) = \{x_o\}$$

by D_1) of Definition 5.1. Thus C_1) holds. Again by (5-1) and (5-2)

$$s_t(\hat{\Psi}(p_o)) = \emptyset \quad \text{if} \quad t < t_o.$$

Finally since $s_{t_1}(\hat{\Psi}(p_o)) = \psi(x_o,t_o,t_1) \neq \emptyset$ by D_2) of Definition 5.1 we have C_2) satisfied.

Now let $\mathcal{B}(\hat{\Psi}) = \{\hat{\Psi}(p_o) : p_o \in X \times R\}$.

Proposition 5.3. $\mathcal{B}(\hat{\Psi})$ is a system of cones defined on $X \times R$ by $\hat{\Psi}$.

Proof: All we need show now is that C_3) of Definition 3.2 holds. Let $p_o \in X \times R$ and $p_1 \in \hat{\Psi}(p_o)$. Suppose $p_2 = (x_2,t_2) \in \hat{\Psi}(p_1)$ then $p_2 \in \Psi(p_1,t_2)$ and $t_1 \leqq t_2$ or, equivalently $x_2 \in \psi(x_1,t_1,t_2)$ and $t_1 \leqq t_2$.

From D_4) of Definition 5.1 we have that:

$$\psi(x_o,t_o,t_2) = \cup \{\psi(x',t',t_2) : x' \in \psi(x_o,t_o,t')\}$$

whenever $t_o \leqq t' \leqq t_2$. Since $p_1 \in \hat{\Psi}(p_o)$ implies that $x_1 \in \psi(x_o,t_o,t_1)$ and $t_o \leqq t_1$, it follows that

$$\psi(x_1,t_1,t_2) \subset \cup \{\psi(x',t_1,t_2) :$$
$$x' \in \psi(x_o,t_o,t_1)\} = \psi(x_o,t_o,t_2)$$

and hence $x_2 \in \psi(x_o,t_o,t_2)$. By (5-1), $(x_2,t_2) = p_2 \in \Psi(p_o,t_2)$ and by (5-2), $p_2 \in \hat{\Psi}(p_o)$. Thus $\hat{\Psi}(p_1) \subset \hat{\Psi}(p_o)$, and C_3) holds.

The question of whether a given system of cones $\mathcal{B}(F)$ on $X \times R$ defines a g.d.s. on X must, in general, be answered in the negative. It is clear that even if each cone $F(p_o)$ is smooth at each $t \geqq t_o$ and some sort of "continuity" condition is placed on F (so that D_5) and D_6) may be satisfied), there remains the problem of satisfying D_3). That this is not possible follows from the fact that $\mathcal{G}(F)$ may be such that for some $p_1 \in X \times R$ $p_1 \notin F(p)$ for all $p \in X \times R$, $p \neq p_1$. In other words, we do not enjoy the luxury of being able to examine the "past" in a system of cones, whereas it is possible to do so in a g.d.s..

We now adapt Roxin's ([9] and [11]) definitions of invariance and strong stability to the setting of the uniform space $(X,\mathcal{U})$ in order to make application of our results.

<u>Definition 5.4</u>. A set $B \subset X$ is called <u>positively strongly invariant</u> with respect to the g.d.s. defined by ψ (positively strongly ψ-invariant) in case $\psi(x_o,t_o,t) \subset B$ for all $x_o \in B$, $t_o \in R$ and $t \geqq t_o$.

<u>Definition 5.5</u>. A positively strongly ψ-invariant set $B \subset X$ is said to be <u>strongly stable</u> with respect to the g.d.s. defined on X by ψ (strongly ψ-stable) in case given $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\psi(x_o,t_o,t) \subset U[B]$ for all $t \geqq t_o$ whenever $x_o \in V[B]$ and

$t_o \in R$.

Recalling Definition 3.12, we will now define an asymptotic stability concept for a g.d.s.. It is a combination of strong stability and uniform quasi-asymptotic strong stability as defined by Roxin [10].

Definition 5.6. A positively strongly ψ-invariant set $B \subset X$ is called strongly asymptotically stable with respect to the g.d.s. defined on X by ψ (strongly asymptotically ψ-stable) provided that B is strongly ψ-stable and there exists a $U \in \mathcal{U}$ such that $\psi(x_o,t_o,t) \to B$ as $t \to \infty$ whenever $x_o \in U[B]$ and $t_o \in R$.

In what follows we will consider a g.d.s. defined on X by ψ and the system of cones $\mathcal{E}(\hat{\Psi})$ generated on $X \times R$ by this g.d.s..

Proposition 5.7. A nonempty set $B \subset X$ is positively strongly ψ-invariant if and only if $A = B \times R$ is $\hat{\Psi}$-invariant.

Proof: Suppose that B is positively strongly ψ-invariant then for any $x_o \in X$, $t_o \in R$, $\psi(x_o,t_o,t) \subset B$ for all $t \geqq t_o$. Let $p_o = (x_o,t_o) \in A$ then $x_o \in B$ and hence $\Psi(p_o,t) = \psi(x_o,t_o,t) \times \{t\} \subset B \times R$ for all $t \geqq t_o$. Thus

$\hat{\Psi}(p_o) = \cup \{\psi(x_o,t_o,t) \times \{t\} : t \geqq t_o\} \subset B \times R = A$ so that A is $\hat{\Psi}$-invariant.

Now suppose that $A = B \times R$ is $\hat{\Psi}$-invariant then if $p_o \in A$, $\hat{\Psi}(p_o) \subset A$. Let $x_o \in B$, $t_o \in R$, and set $p_o = (x_o,t_o)$. We then have $\psi(x_o,t_o,t) = s_t(\hat{\Psi}(p_o)) \subset s_t(A) = s_t(B \times R) = B$ for all $t \geqq t_o$ so that B is strongly ψ-invariant.

Proposition 5.8. A nonempty strongly positively ψ-invariant set $B \subset X$ is strongly ψ-stable if and only if $A = B \times R$ is stable $(\hat{\Psi},\mathcal{W})$.

Proof: Suppose that B is strongly positively ψ-invariant and strongly ψ-stable. Let $W \in \mathcal{W}$ be given. By Lemma 2.11 there is a $U \in \mathcal{U}$ and a $V_\delta \in \mathcal{R}$ such that $S(U,V_\delta) \subset W$. (Of course, $S(U,V_\delta) \in \mathcal{W}$). Corresponding to $U \in \mathcal{U}$ there is a $U_1 \in \mathcal{U}$ such that $\psi(x_o,t_o,t) \subset U[B]$ for all $t \geqq t_o$ whenever $t_o \in R$ and $x_o \in U_1[B]$ by definition of strong ψ-stability. Let $W_1 = S(U_1,V_\delta)$ and suppose that $p_o = (x_o,t_o) \in W_1[A]$. By Lemma 2.25 $W_1[A] = U_1[B] \times R$ so we have $x_o \in U_1[B]$, $t_o \in R$ and, thus, $\psi(x_o,t_o,t) \subset U[B]$ for all $t \geqq t_o$. Therefore,
$\hat{\Psi}(p_o) = \cup\{(\psi(x_o,t_o,t),t) : t \geqq t_o\} \subset U[B] \times R = S(U,V_\delta)[A]$ or $\hat{\Psi}(p_o) \subset W[A]$. Now $p_o \in W_1[A]$ is arbitrary and hence,

A is stable $(\hat{\Psi},\mathcal{W})$.

Conversely suppose that B is strongly positively ψ-invariant and $A = B \times R$ is stable $(\hat{\Psi},\mathcal{W})$. Given $U \in \mathcal{U}$, set $W = S(U, R \times R)$. By the stability of A there is a $W_1 \in \mathcal{W}$ such that $\hat{\Psi}(p_o) \subset W[A]$ whenever $p_o \in W_1[A]$. By Lemma 2.11 there is a $U_1 \in \mathcal{U}$ and $V_\delta \in \mathcal{R}$ such that $S(U_1, V_\delta) \subset W_1$.

Let $x_o \in U_1[B]$ and $t_o \in R$ then $p_o = (x_o, t_o) \in U_1[B] \times R$ and hence by Lemma 2.25 $p_o \in S(U_1, V_\delta)[B \times R] \subset W_1[A]$. Therefore, $\hat{\Psi}(p_o) \subset W[A] = S(U, R \times R)[A] = U[B] \times R$, and it follows that $\psi(x_o, t_o, t) = s_t(\hat{\Psi}(p_o)) \subset U[B]$ for all $t \geqq t_o$. Thus B is strongly ψ-stable.

<u>Proposition 5.9</u>. A non-empty strongly positively ψ-invariant set $B \subset X$ is strongly asymptotically ψ-stable if and only if $A = B \times R$ is uniformly asymptotically stable $(\hat{\Psi},\mathcal{W})$.

Proof: All that remains for us to show is that the condition that there exists a $U \in \mathcal{U}$ such that $\psi(x_o, t_o, t) \to B$ as $t \to \infty$ whenever $x_o \in U[B]$ and $t_o \in R$ is equivalent to A being u.q.a.s.$(\hat{\Psi},\mathcal{W})$. This follows since Proposition 5.8 has already given the equivalence of strong ψ-stability for B and stability $(\hat{\Psi},\mathcal{W})$ for A.

Suppose that there is a $U \in \mathcal{U}$ such that $\psi(x_o,t_o,t) \to B$ as $t \to \infty$ whenever $x_o \in U[B]$ and $t_o \in R$. Recalling Definition 3.9 we have that for any $U_1 \in \mathcal{U}$ there exists $T(U_1) \in R$ such that $\psi(x_o,t_o,t) \subset U_1[B]$ for all $t \geqq t_o + T(U_1)$ whenever $x_o \in U[B]$ and $t_o \in R$. Let $W = S(U, R \times R)$ and let $W_1 \in \mathcal{W}$. By Lemma 2.11 there is a $U_1 \in \mathcal{U}$ and $V_\delta \in \mathcal{R}$ such that $S(U_1,V_\delta) \subset W_1$. Now suppose $p_o = (x_o,t_o) \in W[A] = U[B] \times R$ then $x_o \in U[B]$. Thus there is a $T(U_1) \in R$ such that $\psi(x_o,t_o,t) \subset U_1[B]$ whenever $t \geqq t_o + T(U_1)$ and it follows

$\hat{\Psi}_\tau(p_o) = \cup \{\psi(x_o,t_o,t),t) : t \geqq \tau\} \subset U_1[B] \times R \subset S(U_1,V_\delta)[A]$

for $\tau \geqq t_o + T(U_1)$. Hence $\hat{\Psi}_\tau(p_o) \subset W_1[A]$ whenever $\tau \geqq t_o + T(U_1)$. Since $W_1 \in \mathcal{W}$ is arbitrary we have $\hat{\Psi}_t(p_o) \to A$ as $t \to \infty$. But this holds for each $p_o \in W[A]$ so that by Definition 3.11, A is u.q.a.s.$(\hat{\Psi},\mathcal{W})$.

Conversely suppose $A = B \times R$ is u.q.a.s.$(\hat{\Psi},\mathcal{W})$ then there is a $W \in \mathcal{W}$ such that $\hat{\Psi}_t(p_o) \to A$ as $t \to \infty$ whenever $p_o \in W[A]$. By Lemma 2.11 there exists a $U \in \mathcal{U}$ and $V_\delta \in \mathcal{R}$ such that $S(U,V_\delta) \subset W$. Suppose that

$p_o = (x_o,t_o) \in S(U,V_\delta)[A] = U[B] \times R$ and that $U_1 \in \mathcal{U}$ is given. Let $W_1 = S(U_1, R \times R)$. There exists $T(W_1) \in R$ such that $\hat{\Psi}_t(p_o) \subset W_1[A]$ whenever $t \geqq t_o + T[W_1]$. Hence

$\hat{\Psi}_t(p_o) \subset S(U_1, R \times R)[A] = U_1[B] \times R$ for $t \geqq t_o + T[W_1]$ or we have $s_\tau(\hat{\Psi}_t(p_o)) = \psi(x_o, t_o, \tau) \subset U_1[B]$ for $\tau \geqq t \geqq t_o + T[W_1]$. Thus, if $x_o \in U[B]$, $\psi(x_o, t_o, t) \subset U_1[B]$ for $t \geqq t_o + T[W_1]$ so that $\psi(x_o, t_o, t) \to B$ as $t \to \infty$ since $U_1 \in \mathcal{U}$ is arbitrary.

Finally, we note that if the open set O_A in L_4) of Definition 3.23 is instead a uniform neighborhood of A (that is there is a $W \in \mathcal{W}$ such that $W[A] \subset O_A$), we get a stronger result; namely that A is u.q.a.s.$(F, \mathcal{W})$, rather than q.a.s.$(F, \mathcal{W})$. Hence, we have the following result.

<u>Corollary 5.10</u>. A strongly positively ψ-invariant set $B \subset X$ is strongly asymptotically ψ-stable if and only if there exists a Lyapunov function for $A = B \times R$ into the retracted scale of $\mathcal{W}$ satisfying the condition:

L_4') There is a $W \in \mathcal{W}$ such that if $p_1 \in W[A]$ and $\lambda \in \mathcal{P}_r(\mathcal{W}) \setminus \{0\}$ then there is a $T(\lambda, p_1) \in R$ such that $L(p_1, t) \ngeqq \lambda$ whenever $t > t_1 + T(\lambda, p_1)$.

Proof: The proof is immediate from the remark above and Propositions 3.25, 3.26, and 5.9.

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